MATH 108 Winter 2019 - **Problem Set 1**

due January 18

1. Determine if each propositional form is a tautology, a contradiction, or neither.
   (a) $P \Leftrightarrow P \wedge (P \vee Q)$.
   (b) $[Q \wedge (P \Rightarrow Q)] \Rightarrow P$.
   (c) $P \wedge (P \Leftrightarrow Q) \wedge \sim Q$.
   (d) $(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$.

2. Rewrite each proposition in English. Take the universe to be all real numbers.
   (a) $(\forall x)(\forall y)[(xy > 0) \vee (xy < 0)]$.
   (b) $(\exists x)(\forall y)(x + y = 0)$.
   (c) $(\exists x)(\exists y)(x^2 + y^2 = -1)$.
   (d) $(\forall x)[x > 0 \Rightarrow (\exists y)(xy = 1)]$.
   (e) $(\forall y)(\exists! x)[(x \leq y) \wedge (y \leq x)]$.
   (f) $(\forall y)(\exists! x)(x = y^2)$.

3. Determine if each proposition in Problem 2 is true or false in the universe of all real numbers.

4. Let x be a real number. For each proposition, write the contrapositive. Then prove the proposition by contraposition.
   (a) If $x^2 + 2x < 0$, then $x < 0$.
   (b) If $x(x - 4) > -3$, then $x < 1$ or $x > 3$.

5. Let x and y be real numbers. The "arithmetic-mean and geometric-mean" (AM-GM) inequality is the proposition that if x and y are both nonnegative then
$$\frac{x+y}{2} \geq \sqrt{xy}.$$
   (a) Prove the AM-GM inequality.
   (b) Write the converse of the above statement of the AM-GM inequality.
   (c) Is the converse true? Prove it or give a counterexample.

6. Let a and b be positive integers. Prove each proposition by contradiction.
   (a) If a divides b, then $a \leq b$.
   (b) Either a and b are odd, or ab is even.
   (c) If $a < b$ and $ab < 4$, then $a = 1$.

7. Let x, y and z be three real in the interval $[0, 1]$. Prove that at least two of the numbers have distance $\leq 1/2$ between each other.