

MATH 108 Winter 2019 - Problem Set 1 solutions

due January 18

1. Determine if each propositional form is a tautology, a contradiction, or neither.

(a) $P \Leftrightarrow P \wedge (P \vee Q)$.

P	Q	$P \vee Q$	$P \wedge (P \vee Q)$	$P \Leftrightarrow P \wedge (P \vee Q)$
T	T	T	T	T
T	F	T	T	T
F	T	T	F	T
F	F	F	F	T

Tautology.

(b) $[Q \wedge (P \Rightarrow Q)] \Rightarrow P$.

P	Q	$P \Rightarrow Q$	$Q \wedge (P \Rightarrow Q)$	$[Q \wedge (P \Rightarrow Q)] \Rightarrow P$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Neither.

(c) $P \wedge (P \Leftrightarrow Q) \wedge \sim Q$.

P	Q	$P \Leftrightarrow Q$	$\sim Q$	$P \wedge (P \Leftrightarrow Q) \wedge \sim Q$
T	T	T	F	F
T	F	F	T	F
F	T	F	F	F
F	F	T	T	F

Contradiction.

(d) $(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Neither.

2. Rewrite each proposition in English. Take the universe to be all real numbers.

(a) $(\forall x)(\forall y)[(xy > 0) \vee (xy < 0)]$.

For all real numbers x and y , $xy > 0$ or $xy < 0$.

(b) $(\exists x)(\forall y)(x + y = 0)$.

There is a real number x such that for all real numbers y , $x + y = 0$.

(c) $(\exists x)(\exists y)(x^2 + y^2 = -1)$.

There exist real numbers x and y such that $x^2 + y^2 = -1$.

(d) $(\forall x)[x > 0 \Rightarrow (\exists y)(xy = 1)]$.

For all real numbers x , if $x > 0$ then there is a real number y for which $xy = 1$.

(e) $(\forall y)(\exists!x)[(x \leq y) \wedge (y \leq x)]$.

For each real number y , there is a unique real number x with $x \leq y$ and $y \leq x$.

(f) $(\forall y)(\exists!x)(x = y^2)$.

For all numbers y , there is a unique real number x such that $x = y^2$.

3. Determine if each proposition in Problem 2 is true or false in the universe of all real numbers.

(a) False. If $x = 0$ then neither $xy > 0$ nor $xy < 0$ are true.

(b) False. For any choice of x , there is only one value of y such that $x + y = 0$. For all the other real numbers y , that equation is false.

(c) False. For any choice of real numbers x and y , $x^2 + y^2 \geq 0$, so it cannot be equal to -1 .

(d) True. For any $x > 0$, choosing $y = 1/x$ satisfies $xy = 1$.

(e) True. For any choice of y , there is exactly one value of x for which both $x \leq y$ and $y \leq x$, which is $x = y$.

(f) True. For any choice of y , there is exactly one value of x that is equal to y^2 .

4. Let x be a real number. For each proposition, write the contrapositive. Then prove the proposition by contraposition.

(a) If $x^2 + 2x < 0$, then $x < 0$.

Contrapositive: If $x \geq 0$, then $x^2 + 2x \geq 0$.

Assume that $x \geq 0$. Then we have that $2x \geq 0$. Additionally, $x^2 \geq 0$ for any real x . Summing these two inequalities gives $x^2 + 2x \geq 0$.

(b) If $x(x - 4) > -3$, then $x < 1$ or $x > 3$.

Contrapositive: If $x \geq 1$ and $x \leq 3$, then $x(x - 4) \leq -3$.

Assume that $1 \leq x \leq 3$. Then $x - 1 \geq 0$ and $x - 3 \leq 0$. Since $x - 1$ is nonnegative, we can multiply both sides of the other inequality by $x - 1$ to get

$$(x - 3)(x - 1) \leq 0 \cdot (x - 1) = 0.$$

Rearranging terms of this inequality gives $x(x - 4) \leq -3$.

5. Let x and y be real numbers. The “arithmetic-mean and geometric-mean” (AM-GM) inequality is the proposition that if x and y are both nonnegative then

$$\frac{x + y}{2} \geq \sqrt{xy}.$$

- (a) Prove the AM-GM inequality.

We have that $(x - y)^2 \geq 0$ for all real numbers x and y . Multiplying this out,

$$x^2 - 2xy + y^2 \geq 0,$$

$$x^2 + 2xy + y^2 \geq 4xy,$$

$$\left(\frac{x + y}{2}\right)^2 \geq xy.$$

Since x and y are nonnegative, both sides of the above inequality are nonnegative, so we can take the square root of both sides and preserve the inequality,

$$\sqrt{\left(\frac{x + y}{2}\right)^2} \geq \sqrt{xy}.$$

Because $x + y$ is nonnegative, the left side is equal to $(x + y)/2$. So we have

$$\frac{x + y}{2} \geq \sqrt{xy}.$$

- (b) Write the converse of the above statement of the AM-GM inequality.

Converse: If x and y satisfy $\frac{x+y}{2} \geq \sqrt{xy}$ then $x \geq 0$ and $y \geq 0$.

- (c) Is the converse true? Prove it or give a counterexample.

We prove the converse by contraposition. Suppose that either $x < 0$ or $y < 0$. Without loss of generality, we can assume $x < 0$ since the inequality is symmetric in x and y . Then we consider two cases: either $y > 0$ or $y \leq 0$.

First assume that $y > 0$. Then $xy < 0$ so \sqrt{xy} is not defined over the reals, and so

$$\frac{x + y}{2} \not\geq \sqrt{xy}.$$

Next assume that $y \leq 0$. Then $(x + y)/2 < 0$, but $\sqrt{xy} \geq 0$. Therefore

$$\frac{x + y}{2} < \sqrt{xy}.$$

In both cases, the AM-GM inequality is not satisfied.

6. Let a and b be positive integers. Prove each proposition by contradiction.

- (a) If a divides b , then $a \leq b$.

Assume that a divides b and that $a > b$. By the definition of “divides”, we have that b/a is an integer. Since a and b are positive, b/a must be positive, so $b/a \geq 1$. However, dividing both sides of the inequality $a > b$ by a gives $1 > b/a$. This is a contradiction.

(b) Either a and b are odd, or ab is even.

Assume that a or b are even and that ab is odd. First consider the case that a is even, so $a = 2k$ for some integer k . Then $ab = 2kb$, which is even. This is a contradiction. Otherwise b must be even, so $b = 2\ell$ for some integer ℓ . Then $ab = a2\ell$, which is even. This is also a contradiction. (You could also use “without loss of generality” to reduce cases here.)

(c) If $a < b$ and $ab < 4$, then $a = 1$.

Assume that $a < b$ and $ab < 4$ but $a \neq 1$. Since a is a positive integer, this implies that $a \geq 2$ and since $b > a$, we must have $b \geq 3$. Multiplying these inequalities gives $ab \geq 6$, which is a contradiction.

7. Let x , y and z be three real in the interval $[0, 1]$. Prove that at least two of the numbers have distance $\leq 1/2$ between each other.

Without loss of generality we can assume that $0 \leq x \leq y \leq z \leq 1$ by relabelling the numbers so that they are in order. We proceed by contradiction. Assume that no pair has distance $\leq 1/2$. Then $y - x > 1/2$ and $z - y > 1/2$. Rewrite these as $y > x + 1/2$ and $z > y + 1/2$. Since $x \geq 0$, the first inequality gives $y > 1/2$. Combining this with the second inequality gives

$$z > 1/2 + 1/2 = 1.$$

But this contradicts the fact that $z \leq 1$. Therefore there must be a pair with distance $\leq 1/2$.