MATH 108 Winter 2019 - **Problem Set 2 solutions**

due January 25

1. (a) Prove that there exist integers m and n such that $3m + 4n = 1$.

   Let $m = -1$ and $n = 1$. Then $3m + 4n = 3(-1) + 4(1) = 1$.

   (b) Prove that there does not exist integers m and n such that $3m + 6n = 1$.

   Since $3m + 6n = 3(m + 2n)$ and $m + 2n$ is an integer, $3m + 6n$ is divisible by 3 for all integers m and n. On the other hand 1 is not divisible by 3. Therefore $3m + 6n$ cannot be equal to 1.

2. For all integers x, prove that x is divisible by 6 if and only if x is divisible by both 2 and 3.

   Assume that x is divisible by 6 so $x = 6k$ for some integr k. Then $x = 2(3k)$ so x is divisible by 2, and $x = 3(2k)$ so x is divisible by 3.

   Assume that x is divisible by 2 and 3. Since x is divisible by 3, $x = 3k$ for some integer k. Since x is even, either 3 is even or k is even. But 3 is not even, so k is even. Therefore $k = 2\ell$ for some integer ℓ. Then $x = 3k = 6\ell$, so it is divisible by 6.

3. Let $A = \{1, 2\}$ and $B = \{1, 4, 5\}$.

   (a) Find $A \cup B$.

   $A \cup B = \{1, 2, 4, 5\}$.

   (b) Find $A \times B$.

   $A \times B = \{(1, 1), (1, 4), (1, 5), (2, 1), (2, 4), (2, 5)\}$.

   (c) Find $\mathcal{P}(A)$.

   $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

4. Let A, B, C, D be sets. For each proposition, give a proof or a counterexample.

   (a) $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

   True. Suppose $x \in (A \cup B) \cap C$. Then $x \in A \cup B$ and $x \in C$. Then $x \in A$ and $x \in C$, or $x \in B$ and $x \in C$. We consider each case.

   Assume that $x \in A$ and $x \in C$. Since $x \in A$ and $A \subseteq A \cup (B \cap C)$, we have $x \in A \cup (B \cap C)$.

   Assume that $x \in B$ and $x \in C$. Then $x \in B \cap C$ and $B \cap C \subseteq A \cup (B \cap C)$, we have $x \in A \cup (B \cap C)$.

   (b) If $A \cap C \subseteq B \cap C$, then $A \subseteq B$.

   False. Let $A = \{0\}$ and $B = C = \emptyset$. Then $A \cap C = B \cap C = \emptyset$, so $A \cap C \subseteq B \cap C$, but $A \not\subseteq B$.

(c) $\mathcal{P}(A) \setminus \mathcal{P}(B) \subseteq \mathcal{P}(A \setminus B)$.
False. Let $A = \{0, 1\}$ and $B = \{0\}$. Then $\mathcal{P}(A) \setminus \mathcal{P}(B) = \{\{1\}, \{0, 1\}\}$ and $\mathcal{P}(A \setminus B) = \{\emptyset, \{1\}\}$. So $\mathcal{P}(A) \setminus \mathcal{P}(B) \not\subseteq \mathcal{P}(A \setminus B)$.

(d) If A and B are disjoint, then $A \cap C$ and $B \cap C$ are disjoint.
True. Suppose A and B are disjoint, so $A \cap B = \emptyset$. Then

$$(A \cap C) \cap (B \cap C) = A \cap B \cap C \subseteq A \cap B = \emptyset.$$

So then $(A \cap C) \cap (B \cap C) = \emptyset$, which means $A \cap C$ and $B \cap C$ are disjoint.

(e) If $C \subseteq A$ and $D \subseteq B$ then $D \setminus A \subseteq B \setminus C$.
True. Assume that $C \subseteq A$ and $D \subseteq B$. Suppose $x \in D \setminus A$, so $x \in D$ and $x \notin A$. We have from $D \subseteq B$ that if $x \in D$ then $x \in B$, so we can conclude that $x \in B$. We have from $C \subseteq A$ that if $x \in C$ then $x \in A$. The contrapositive of this statement combined with $x \notin A$ implies $x \notin C$. Therefore $x \in B \setminus C$.

(f) If $A \cap B \cap C = \emptyset$, then A, B, C are pair-wise disjoint.
False. Let $A = B = \{0\}$ and $C = \emptyset$. Then $A \cap B \cap C = \emptyset$. However $A \cap B = \{0\}$, so they are not disjoint.

5. Let A be the set of positive integers that are not perfect squares. Let P be the set of prime numbers. Prove that $P \subseteq A$.

We proceed by contradiction. Assume that $P \not\subseteq A$, so there exists some $p \in P$ such that $p \notin A$. Since $p \notin A$, it is a square, so $p = n^2$ for some integer n. Since p is a prime, $p > 1$, so $n > 1$ as well. However $p = n \cdot n$ with $n > 1$ implies that p is composite. This contradicts $p \in P$.

6. (a) Prove that if U and V are finite sets then

$$|U| + |V| = |U \cup V| + |U \cap V|.$$

$U \cup V$ can be broken up into three disjoin pieces, $U \setminus V$, $V \setminus U$ and $U \cap V$. Let $|U \setminus V| = a$, $|V \setminus U| = b$ and $|U \cap V| = c$. Therefore $|U \cup V| = a + b + c$. The set U is the disjoint union of $U \setminus V$ and $U \cap V$, so $|U| = a + c$. Similarly V is the disjoint union of $V \setminus U$ and $U \cap V$, so $|U| = b + c$. Then

$$|U| + |V| = (a + c) + (b + c) = a + b + 2c,$$

$$|U \cup V| + |U \cap V| = (a + b + c) + c = a + b + 2c.$$

(b) Prove that if U and V are finite sets then $U \cup V$ is finite.
Assume that U and V are finite sets, so $|U| + |V|$ is finite. Then from part (a)

$$|U \cup V| = |U| + |V| - |U \cap V|.$$

We know that $|U \cap V| \geq 0$, so then

$$|U \cup V| \leq |U| + |V| < \infty,$$

so $U \cup V$ is finite.

7. Prove by example that there exist sets A and B with $A \subsetneq B$ and a function $f : B \to A$ that is injective (1-to-1).

   Let $B = \mathbb{Z}$ and $A = 2\mathbb{Z}$, and define $f : B \to A$ to be $f(x) = 2x$. If x and y are integers with $x \neq y$, then $2x \neq 2y$ so $f(x) \neq f(y)$. This proves that f is injective. If x is an integer, then $2x$ is an integer, so $A \subseteq B$. We have $1 \in B$ but 1 is not even, so $1 \notin A$. This proves that $A \subsetneq B$.