MATH 108 Winter 2019 - **Problem Set 3 solutions**

due February 1

1. Using induction, prove that for all positive integers n,

   (a) $n^3 - n$ is divisible by 3.
   The base case is $n = 1$. $1^3 - 1 = 0$ which is divisible by 3.
   Assume for some $n \geq 1$ that $n^3 - n$ is divisible by 3. Then

   $$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1$$
   $$= (n^3 - n) + 3n^2 + 3n.$$

   Since $3n^2$ and $3n$ are multiples of 3 and $n^3 - n$ is divisible by 3, the sum is also divisible by 3.

   (b) $8^n - 1$ is divisible by 7.
   The base case is $n = 1$. $8^1 - 1 = 7$ which is divisible by 7.
   Assume for some $n \geq 1$ that $8^n - 1$ is divisible by 7. Then

   $$8^{n+1} - 1 = 8 \cdot 8^n - 1$$
   $$= 8 \cdot 8^n - 8 + 7 = 8(8^n - 1) + 7.$$

   Since $8(8^n - 1)$ is divisible by $8^n - 1$, it is divisible by 7. Clearly 7 is also divisible by 7, so the sum is divisible by 7.

   (c) $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.
   The base case is $n = 1$. $\sum_{k=1}^{1} k^3 = 1^3 = 1$. On the other side, $\frac{1^2(1+1)^2}{4} = 1$, so the equality holds.
   Assume for some $n \geq 1$ that $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$. Then

   $$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3.$$

   By the induction hypothesis,

   $$= \frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{(n^4 + 2n^3 + n^2) + (4n^3 + 12n^2 + 12n + 4)}{4}$$
   $$= \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4}.$$

   On the other side,

   $$\frac{(n+1)^2(n+2)^2}{4} = \frac{(n^2 + 2n + 1)(n^2 + 4n + 4)}{4}$$
   $$= \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4}$$

   so the equality holds for $n + 1$.

(d) $n! = 1 + \sum_{k=1}^{n-1} k \cdot k!$.

The base case is $n = 1$. $1! = 1$. On the other side, $1 + \sum_{k=1}^{0} k \cdot k! = 1 + 0$, so the equality holds.

Assume for some $n \geq 1$ that $n! = 1 + \sum_{k=1}^{n-1} k \cdot k!$. Then

$$1 + \sum_{k=1}^{n} k \cdot k! = 1 + \sum_{k=1}^{n-1} k \cdot k! + n \cdot n!$$

By the induction hypothesis,

$$= n! + n \cdot n! = (n+1)n! = (n+1)!.$$

2. In American football, a team can score seven points for each touchdown, and three points for each field goal (ignore safeties, two-point conversions, etc). Prove that every integer score larger than 11 is possible.

   The base cases are $n = 12, 13, 14$. 12 points can be obtained from 4 field goals. 13 points can be obtained from a touchdown and two field goals. 14 points can be obtained from two touchdowns.

   For $n \geq 15$ assume that $n - 3$ points is possible, so $n - 3 = 3a + 7b$ for some nonnegative integers a and b. then

$$n = 3(a+1) + 7b$$

   so n points can be obtained by $a + 1$ field goals and b touchdowns.

3. Let P be the set of prime numbers. Prove that

$$\bigcup_{p \in P} p\mathbb{Z} = \mathbb{Z} \setminus \{-1, 1\}.$$

   Assume that $x \in \bigcup_{p \in P} p\mathbb{Z}$, so $x \in p\mathbb{Z}$ for some prime p. Since $p\mathbb{Z} \subseteq \mathbb{Z}$, x is an integer. However 1 and -1 are not divisible by any prime p, since $p \geq 2$, so $x \neq 1$ and $x \neq -1$. Therefore $x \in \mathbb{Z} \setminus \{-1, 1\}$. This proves $\bigcup_{p \in P} p\mathbb{Z} \subseteq \mathbb{Z} \setminus \{-1, 1\}$.

   Assume that $x \in \mathbb{Z} \setminus \{-1, 1\}$. We consider three cases. First assume x is positive, so $x \geq 2$. By the theorem showed in class, x is divisible by a prime number p, so $x \in p\mathbb{Z} \subseteq \bigcup_{p \in P} p\mathbb{Z}$. Next assume x is negative, so $x \leq -2$. Then $-x$ is divisible by a prime p by the theorem. So $-x = kp$ for some integer k. Then $x = (-k)p$ so $x \in p\mathbb{Z} \subseteq \bigcup_{p \in P} p\mathbb{Z}$. Finally assume $x = 0$. Then for any prime p, $x = 0 \cdot p$, so $x \in p\mathbb{Z} \subseteq \bigcup_{p \in P} p\mathbb{Z}$. This proves $\bigcup_{p \in P} p\mathbb{Z} \supseteq \mathbb{Z} \setminus \{-1, 1\}$.

4. Using induction, prove that if A is a finite set with $|A| = n$ then $|\mathcal{P}(A)| = 2^n$.

   The base case is $n = 0$. If $A = \emptyset$ then $\mathcal{P}(A) = \{\emptyset\}$ so $|\mathcal{P}(A)| = 1 = 2^0$.

   Assume for some $n \geq 0$ that for any set A if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$. Let B be a set with size $n + 1$. Then there is some element $x \in B$. Let $A = B \setminus \{x\}$ so $|A| = n$. We can break up $\mathcal{P}(B)$ into two disjoint pieces,

$$\mathcal{P}(B) = \{S \in \mathcal{P}(B) \mid x \notin S\} \cup \{S \in \mathcal{P}(B) \mid x \in S\}.$$

Denote these subsets C and D. The subsets of B that do not contain x are exactly the subsets of A. Therefore $C = \mathcal{P}(A)$, and so $|C| = 2^n$ by the induction hypothesis. For every set $S \in C$, there is set $S \cup \{x\} \in D$. Similarly for every set $T \in D$, there is set $T \setminus \{x\} \in C$. Therefore $|D| = |C| = 2^n$. Since $\mathcal{P}(B)$ is the disjoint union of C and D, we have

$$|\mathcal{P}(B)| = |C| + |D| = 2^n + 2^n = 2^{n+1}.$$

5. Use the Well-Ordering Principle of the natural numbers to prove that every positive rational number x can be expressed as a fraction $x = a/b$ where a and b are positive integers with no common factor.

   Let
   $$S = \{a \in \mathbb{N}_1 \mid \exists b \in \mathbb{Z} \text{ s.t. } x = a/b\}.$$

   Since x is a rational number, it can be expressed as a fraction of integers $x = c/d$. Since x is positive, either c is positive, so $c \in S$, or else c is negative and $x = (-c)/(-d)$, so $-c \in S$. Therefore S is not empty. By the Well-Ordering Principle, S has a smallest element, a.

   Suppose that $x = a/b$ and that a and b have a common factor $d > 1$. Then $a = dk$ and $b = d\ell$ for some integers k and ℓ. Note then that $0 < k < a$. We have $x = (dk)/(d\ell) = k/\ell$ with $k > 0$ so then $k \in S$. But this contradicts the fact that a was the smallest element of S. Therefore it must be that a and b have no common factor. Finally, since x and a are positive, b must also be positive.

6. The Fibonacci sequence is an infinite sequence of integers $(f_0, f_1, f_2, f_3, \ldots)$ defined as follows. The first two numbers are $f_0 = 0$ and $f_1 = 1$. For all $n \geq 2$, define f_n to be the sum of the previous two numbers,

   $$f_n = f_{n-1} + f_{n-2}.$$

   Use induction to prove that for all nonnegative integers n,

   $$f_n = \frac{\varphi^n - \psi^n}{\varphi - \psi},$$

   where $\varphi = (1 + \sqrt{5})/2$ and $\psi = (1 - \sqrt{5})/2$.

   The base cases are $n = 0, 1$. For $n = 0$,

   $$\frac{\varphi^0 - \psi^0}{\varphi - \psi} = \frac{0}{\varphi - \psi} = 0 = f_0.$$

   For $n = 1$,

   $$\frac{\varphi^1 - \psi^1}{\varphi - \psi} = \frac{\varphi - \psi}{\varphi - \psi} = 1 = f_1.$$

   Assume for some $n \geq 2$ that $f_k = \frac{\varphi^k - \psi^k}{\varphi - \psi}$ for all $0 \leq k < n$. Then

   $$f_n = f_{n-1} + f_{n-2}$$

and by the induction hypothesis

$$= \frac{\varphi^{n-1} - \psi^{n-1}}{\varphi - \psi} + \frac{\varphi^{n-2} - \psi^{n-2}}{\varphi - \psi} = \frac{(\varphi^{n-1} + \varphi^{n-2}) - (\psi^{n-1} + \psi^{n-2})}{\varphi - \psi}.$$

We have

$$\varphi^{n-1} + \varphi^{n-2} = (\varphi + 1)\varphi^{n-2} = \frac{3 + \sqrt{5}}{2}\varphi^{n-2}$$

$$= \left(\frac{1 + \sqrt{5}}{2}\right)^2 \varphi^{n-2} = \varphi^2 \cdot \varphi^{n-2} = \varphi^n.$$

Similarly

$$\psi^{n-1} + \psi^{n-2} = (\psi + 1)\psi^{n-2} = \frac{3 - \sqrt{5}}{2}\psi^{n-2}$$

$$= \left(\frac{1 - \sqrt{5}}{2}\right)^2 \varphi^{n-2} = \psi^2 \cdot \psi^{n-2} = \psi^n.$$

Therefore

$$f_n = \frac{\varphi^n - \psi^n}{\varphi - \psi}.$$

7. For each positive integer n, let T_n count the number of ways to tile a $2 \times n$ rectangle using 2×1 tiles. Prove for each positive integer n that $T_n = f_{n+1}$, the $(n+1)$th Fibonacci number.

   The bases cases are $n = 1, 2$. For a 2×1 rectangle, there is only one tiling, using one tile, so $T_1 = 1 = f_2$. For a 2×2 rectangle, there are two tilings, using two horizontal tiles or two vertical tiles, so $T_2 = 2 = f_3$.

   Assume for some $n \geq 3$ that $T_k = f_{k+1}$ for all $0 \leq k < n$. To tile a $2 \times n$ rectangle, consider how we can start the tiling from the left edge. To cover the two squares in the left most column, there are two possible choices: either use one vertical tile, or two horizontal tiles. In the first case, the remaining untiled space is a $2 \times (n-1)$ rectangle, so there are T_{n-1} ways to tile the remaining space. That gives T_{n-1} ways to tile the full $2 \times n$ rectangle. In the second case, the remaining untiled space is a $2 \times (n-2)$ rectangle, so there are T_{n-2} ways to tile the remaining space. That gives T_{n-2} ways to tile the full $2 \times n$ rectangle. These two cases cover all possible ways, so

$$T_n = T_{n-1} + T_{n-2}.$$

   By the induction hypothesis,

$$T_{n-1} + T_{n-2} = f_n + f_{n-1}$$

   and by the definition of the Fibonacci numbers, this is equal to f_{n+1}.