MATH 108 Winter 2019 - **Problem Set 6**

due March 1

1. Prove that each function is a bijection.

   (a) $f : (2, \infty) \to (-\infty, -1)$ defined by $f(x) = \dfrac{-x}{x-2}$.

   (b) $f : \mathbb{N}_1 \times \mathbb{N}_1 \to \mathbb{N}_1$ defined by $f(x, y) = 2^{x-1}(2y - 1)$.

   (c) $f : \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$ defined by $f(\overline{x}) = \overline{5x - 1}$.

2. For postive integers n and m, let $[n] = \{1, 2, \ldots, n\}$ and $[m] = \{1, 2, \ldots, m\}$.

   (a) Let A be the set of all functions from $[n]$ to $[m]$. Compute $|A|$ in terms of n and m.

   (b) Let B be the set of all bijective functions from $[n]$ to $[m]$. Compute $|B|$ in terms of n and m.

   (c) Let C be the set of all injective functions from $[n]$ to $[m]$. Compute $|C|$ in terms of n and m.

3. Let A and B be finite sets with $|A| = |B|$, and let $f : A \to B$.

   (a) Prove that if f is injective, then f is surjective.

   (b) Prove that if f is surjective, then f is injective.

4. Let A be a finite set and B be an infinite set with $A \subseteq B$. Prove that $B \setminus A$ is infinite.

5. For each infinite set, determine if it is countable or uncountable. Then prove your answer.

   (a) The set of prime numbers.

   (b) $\mathbb{N}_1 \times \mathbb{N}_1 \times \mathbb{N}_1$.

   (d) The set of all finite-length binary strings, $\bigcup_{n=0}^{\infty} \{0, 1\}^n$. (This is the set of all possible computer files.)

7. Let $\mathbb{N}_0^\infty$ denote the set of all infinite sequences of nonnegative integers,

$$\mathbb{N}_0^\infty = \{(a_1, a_2, a_3, \ldots) \mid a_1, a_2, a_3, \ldots \in \mathbb{N}_0\}.$$

   Let A be the subset of $\mathbb{N}_0^\infty$ consisting of all sequences that have only a finite number of nonzero entries. Prove that A is countable by finding a bijection between $\mathbb{N}_1$ and A.

   [Hint: For each $n \in \mathbb{N}_1$, use the prime factorization of n to produce a sequence in A.]