MATH 108 Winter 2019 - **Problem Set 6 solutions**

due March 1

1. Prove that each function is a bijection.

   (a) $f : (2, \infty) \to (-\infty, -1)$ defined by $f(x) = \dfrac{-x}{x-2}$.

   For $y \in (-\infty, -1)$, solving $f(x) = y$ for x shows that $x = 2y/(y+1)$ is the unique real number that could map to y. Therefore there is at most one $x \in (2, \infty)$ with $f(x) = y$, proving f is injective.

   To see that f is surjective, we need to check that for each $y \in (-\infty, -1)$, the value $x = 2y/(y+1)$ that would map to y is in the domain, $(2, \infty)$. Since $y < -1$, the denominator $y + 1$ is negative. Dividing both sides of the inequality $y < y + 1$ by $y + 1$ gives

   $$\frac{y}{y+1} > 1.$$

   Therefore $x = 2y/(y + 1) > 2$, which proves that there is $x \in (2, \infty)$ with $f(x) = y$.

   (b) $f : \mathbb{N}_1 \times \mathbb{N}_1 \to \mathbb{N}_1$ defined by $f(x, y) = 2^{x-1}(2y - 1)$.

   For $n \in \mathbb{N}_1$, consider the prime factorization of n. Separating out all the factors of 2, we can express n as $2^k m$ where m is odd and k is nonnegative. Then $m = 2y - 1$ for some postive integer y, and $k = x - 1$ for some positive integer x. Therefore $n = f(x, y)$, proving that f is surjective.

   Let $(x, y), (z, w) \in \mathbb{N}_1 \times \mathbb{N}_1$ with $f(x, y) = f(z, w)$. Without loss of generallity, assume $x \geq z$. Then

   $$2^{x-1}(2y - 1) = 2^{z-1}(2w - 1),$$
   $$2^{x-z}(2y - 1) = 2w - 1.$$

   Since the right side is odd, $x - z = 0$. Then $2y - 1 = 2w - 1$ so $y = w$. Therefore $(x, y) = (z, w)$ proving that f is injective.

   (c) $f : \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$ defined by $f(\overline{x}) = \overline{5x - 1}$.

   $$f(\overline{0}) = \overline{7},$$
   $$f(\overline{1}) = \overline{4},$$
   $$f(\overline{2}) = \overline{1},$$
   $$f(\overline{3}) = \overline{6},$$
   $$f(\overline{4}) = \overline{3},$$
   $$f(\overline{5}) = \overline{0},$$
   $$f(\overline{6}) = \overline{5},$$
   $$f(\overline{7}) = \overline{2}.$$

   For each $\overline{y} \in \mathbb{Z}/8\mathbb{Z}$, there is exactly one $\overline{x} \in \mathbb{Z}/8\mathbb{Z}$ with $f(\overline{x}) = \overline{y}$.

2. For postive integers n and m, let $[n] = \{1, 2, \dots, n\}$ and $[m] = \{1, 2, \dots, m\}$.

(a) Let A be the set of all functions from $[n]$ to $[m]$. Compute $|A|$ in terms of n and m.

For $f : [n] \to [m]$, we can choose the values $f(k)$ one at a time for each k from 1 up to n. For each k there are m choices for $f(k)$, so the total number of functions is m^n.

(b) Let B be the set of all bijective functions from $[n]$ to $[m]$. Compute $|B|$ in terms of n and m.

If $n \neq m$ then any function $f : [n] \to [m]$ can't be bijective, so there are 0 bijective functions.

If $n = m$, we again choose the values of $f(k)$ one at a time for each k from 1 up to n. When $k = 1$, there are n possible values for $f(1)$ to choose from. Once $f(1)$ is chosen, there are only $n - 1$ choices for $f(2)$ because $f(2)$ can't be equal to $f(1)$ if f is injective. Once $f(1)$ and $f(2)$ are chosen, there are $n - 2$ choices left for $f(3)$. This repeats for each k up to $k = n$ where we have only one choice left for $f(n)$. Therefore the number of possible bijective functions is

$$n \cdot (n-1) \cdot (n-2) \cdots 1 = n!.$$

(c) Let C be the set of all injective functions from $[n]$ to $[m]$. Compute $|C|$ in terms of n and m.

If $n > m$ then any function $f : [n] \to [m]$ can't be injective, so there are 0 injective functions.

If $n \leq m$, the analysis is similar to part (b). We can choose the values of $f(k)$ one at a time. The number of choices for $f(1)$ is m, for $f(2)$ is $m - 1$ and so on. The last decision is $f(n)$ which has $m - n + 1$ choices left. Therfore the number of possible injective functions is

$$m \cdot (m-1) \cdot (m-2) \cdots (m-n+1) = \frac{m!}{(m-n)!}.$$

3. Let A and B be finite sets with $|A| = |B|$, and let $f : A \to B$.

(a) Prove that if f is injective, then f is surjective.

We proceed by contradiction. Suppose that f is injective, but not surjective. Since f is not surjective, $\operatorname{Im}(f) \subsetneq B$. Using the fact that B is finite, this means that $|\operatorname{Im}(f)| < |B|$. Since f is injective, f matches up each element of A with a distinct element of $\operatorname{Im}(f)$, so $|A| = |\operatorname{Im}(f)|$. This contradicts the fact that $|A| = |B|$.

(b) Prove that if f is surjective, then f is injective.

Let $c : B \to \mathbb{N}_0$ be a new function with $c(y)$ counting number of elements $x \in A$ with $f(x) = y$. Note that

$$|A| = \sum_{y \in B} c(y)$$

because every element of A will get counted once.

Suppose that f is surjective, but not injective. This means $c(y) \geq 1$ for all $y \in B$, and there is some $z \in B$ with $c(z) > 1$. Therefore

$$|A| = \sum_{y \in B} c(y) > \sum_{y \in B} 1 = |B|.$$

But this contradicts the fact that $|A| = |B|$.

4. Let A be a finite set and B be an infinite set with $A \subseteq B$. Prove that $B \setminus A$ is infinite.

   Suppose $B \setminus A$ is finite. From Problem Set 2, if $B \setminus A$ and A are finite, then their union,

   $$(B \setminus A) \cup A = B$$

   is finite, which is a contradiction. Therefore $B \setminus A$ is infinite.

5. For each infinite set, determine if it is countable or uncountable. Then prove your answer.

   (a) The set of prime numbers.

   Countable. The set of prime numbers is a subset of a countable set $\mathbb{N}_1$, so it is countable.

   (b) $\mathbb{N}_1 \times \mathbb{N}_1 \times \mathbb{N}_1$.

   Countable. We twice apply the fact that the product of two countable sets is countable. Since $\mathbb{N}_1$ is countable, it follows that product $\mathbb{N}_1 \times \mathbb{N}_1$ is countable. Then the product $(\mathbb{N}_1 \times \mathbb{N}_1) \times \mathbb{N}_1$ is countable.

   (d) The set of all finite-length binary strings, $\bigcup_{n=0}^{\infty} \{0,1\}^n$. (This is the set of all possible computer files.)

   Countable. For each nonnegative integer n, the set $\{0,1\}^n$ of binary strings of length n is finite (with size 2^n). We can enumerate the elements of $\bigcup_{n=0}^{\infty} \{0,1\}^n$ by first listing all elements of $\{0,1\}^0$ followed by the elements of $\{0,1\}^1$, then of $\{0,1\}^2$, etc. This description of how to list the elements is sufficient to show that the set is countable.

   It is possible to explicitly describe the bijection $f : \bigcup_{n=0}^{\infty} \{0,1\}^n \to \mathbb{N}_1$ that the above listing process describes. For each binary sequence s, let n be the length of s and let k be the positive integer encoded by s. Then define $f(s) = 2^n + k$.

   Another proof is to give an injective map to $g : \bigcup_{n=0}^{\infty} \{0,1\}^n \to \mathbb{N}_0 \times \mathbb{N}_1$. For each $n \in \mathbb{N}_0$, there is an injection $j_n : \{0,1\}^n \to \mathbb{N}_1$ mapping sequence s to the number that it encodes in binary. Then Define $g(s) = (n, j_n(s))$ where n is the length of s. Suppose $g(s) = g(t) = (n, k)$ for some binary sequences s and t. Both s and t have length n, and they encode the same number k in binary, so $s = t$. This proves that g is injective. Therefore $|\bigcup_{n=0}^{\infty} \{0,1\}^n| \leq |\mathbb{N}_0 \times \mathbb{N}_1| = \aleph_0$.

7. Let A be the subset of $\mathbb{N}_0^{\infty}$ consisting of all sequences that have only a finite number of nonzero entries. Prove that A is countable by finding a bijection between $\mathbb{N}_1$ and A.

   [Hint: For each $n \in \mathbb{N}_1$, use the prime factorization of n to produce a sequence in A.]

Let $p_1, p_2, p_3, \ldots$ be the list of all primes in increasing order (so $p_1 = 2$, $p_2 = 3$, etc). Define $f : A \to \mathbb{N}_1$ by

$$f(a_1, a_2, a_3, \ldots) = \prod_{k=1}^{\infty} p_k^{a_k}.$$

For $f(a_1, a_2, a_3, \ldots)$ to be well-defined, we have to check that the infinite product on the right-hand side converges. For any sequence $(a_1, a_1, a_3, \ldots) \in A$, only a finite number of entries are nonzero. Therefore $p_k^{a_k} = 1$ for all but a finite number of terms. So only a finite number of terms contribute to the product. The product of a finite number of integers is an integer.

Every $n \in \mathbb{N}_1$ has a prime factorization, so f is surjective. Since the prime factorization of $n \in \mathbb{N}_1$ is unique, f is injective.