MATH 108 Winter 2019: Intro to Abstract Math

Midterm topics

Logic and Proofs (Sec 1.1-1.6)

- Propositions
- Logical connectives ($\sim$, $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$)
- Truth tables, tautologies, contradictions
- Contrapositive and converse
- Quantifiers ($\forall$, $\exists$, $\exists!$)
- Direct proofs (for conditionals)
- Proofs by contraposition (for conditionals)
- Proofs by contradiction
- Two-way proofs (for biconditionals)
- Proofs by cases

Sets and Induction (Sec 2.1-2.5)

- Set operations ($\cup$, $\cap$, $\setminus$, $\times$, $\mathcal{P}$)
- Big union and big intersection ($\bigcup$, $\bigcap$)
- Proofs of $A \subseteq B$
- Cardinality (size) of sets
- "Weak" induction proofs (with one or multiple base cases)
- "Strong" induction proofs
- Well-Ordering Principle of $\mathbb{N}_0$
- Euclid's Lemma
- Bézout's Identity
- Fundamental Theorem of Arithmetic

Relations and Partitions (Sec 3.1-3.5)

- Properties of relations (reflexive, irreflexive, symmetric, antisymmetric, transitive)
- Directed graphs
- Equivalence relations
- Equivalence classes, quotients, quotient maps
- Partitions
- Modular arithmetic
- Partial orders
- Hasse diagrams
- Least upper bounds and greatest lower bounds

Practice Problems

1. (a) Write the truth table for the propositional form $(P \Rightarrow Q) \vee (Q \Rightarrow P)$.
   (b) Is $(P \Rightarrow Q) \vee (Q \Rightarrow P)$ a tautology, a contradiction, or neither?

2. Let P be the proposition $(\forall x)(\forall y)((x + y \notin \mathbb{Z}) \Rightarrow (x \notin \mathbb{Z} \vee y \notin \mathbb{Z}))$ with universe $\mathbb{R}$.
   (a) Write P in English.
   (b) Write the contrapositive of P.
   (c) Prove P by contraposition.

3. Let P be the proposition "For all integers n, n is odd or $n + 1$ is odd."
   (a) Write the negation of P.
   (b) Prove P by contradiction.

4. For positive integers a, b, c, prove that ac divides bc if and only if a divides b.

5. For sets A, B, C, prove that $(A \setminus B) \setminus C = (A \setminus C) \setminus (B \setminus C)$.

6. Give an example of sets A, B, C with $A \setminus C \subseteq B \setminus C$ but $A \not\subseteq B$.

7. Prove by induction that $n! \geq 2^{n-1}$ for all positive integers n.

8. Prove by induction that using 2 cent stamps and 5 cent stamps, one can make n cents worth of postage for all $n \geq 4$.

9. Let $\sim$ be the relation on $\mathbb{Z}$ defined by $x \sim y$ if and only if $|x - y| \leq 1$. Which of the following properties does $\sim$ have: reflexive, irreflexive, symmetric, antisymmetric, transitive?

10. Let $\sim$ be the relation on $\mathbb{R}$ defined by $x \sim y$ if and only if $\sin x = \sin y$.
    (a) Prove that $\sim$ is an equivalence relation.
    (b) Describe the equivalence class of 0.

11. Prove with modular arithmetic that the last digit of 9^n is 1 or 9 for all positive integers n.

12. Define relation $\preceq$ on $\mathbb{Z}^2$ by $(a, b) \preceq (c, d)$ if and only if $a \leq c$ and $b \leq d$.
    (a) Prove that $\preceq$ is a partial order.
    (b) Find the greatest lower bound of $\{(1, 5), (3, 3)\}$.
    (c) Is $\preceq$ a total order?