MATH 108 Fall 2019 - **Problem Set 1 solutions**

due October 4

1. Write the truth table for each propositional form, and determine if it is a tautology, a contradiction, or neither.

   (a) $P \Leftrightarrow P \wedge (P \vee Q)$.

P	Q	$P \vee Q$	$P \wedge (P \vee Q)$	$P \Leftrightarrow P \wedge (P \vee Q)$
T	T	T	T	T
T	F	T	T	T
F	T	T	F	T
F	F	F	F	T

   Tautology.

   (b) $[Q \wedge (P \Rightarrow Q)] \Rightarrow P$.

P	Q	$P \Rightarrow Q$	$Q \wedge (P \Rightarrow Q)$	$[Q \wedge (P \Rightarrow Q)] \Rightarrow P$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

   Neither.

   (c) $P \wedge (P \Leftrightarrow Q) \wedge \sim Q$.

P	Q	$P \Leftrightarrow Q$	$\sim Q$	$P \wedge (P \Leftrightarrow Q) \wedge \sim Q$
T	T	T	F	F
T	F	F	T	F
F	T	F	F	F
F	F	T	T	F

   Contradiction.

   (d) $(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

   Neither.

2. Rewrite each proposition in English. You may use mathematical expressions (e.g. "$x = 0$") in your answers but replace all the logical symbols. Take the universe to be all real numbers.

   (a) $(\forall x)(\forall y)[(xy > 0) \vee (xy < 0)]$.

   For all real numbers x and y, $xy > 0$ or $xy < 0$.

(b) $(\exists x)(\forall y)(x + y = 0)$.
There is a real number x such that for all real numbers y, $x + y = 0$.

(c) $(\forall y)(\exists x)(x + y = 0)$.
For all real numbers y, there is a real number x such that $x + y = 0$.

(d) $(\forall x)[x > 0 \Rightarrow (\exists y)(xy = 1)]$.
For all real numbers x, if $x > 0$ then there is a real number y for which $xy = 1$.

(e) $(\forall y)(\exists! x)[(x \leq y) \wedge (y \leq x)]$.
For each real number y, there is a unique real number x with $x \leq y$ and $y \leq x$.

(f) $(\forall y)(\exists! x)(y = x^2)$.
For all numbers y, there is a unique real number x such that $y = x^2$.

3. Determine if each proposition in Problem 2 is true or false in the universe of all real numbers. Give a short justification for each answer.

   (a) False. If $x = 0$ then neither $xy > 0$ nor $xy < 0$ are true.

   (b) False. For any choice of x, there is onle one value of y such that $x + y = 0$. For all the other real numbers y, that equation is false.

   (c) True. For all y, choosing $x = -y$ satisfies $x + y = 0$.

   (d) True. For any $x > 0$, choosing $y = 1/x$ satisfies $xy = 1$.

   (e) True. For any choice of y, there is exactly one value of x for which both $x \leq y$ and $y \leq x$, which is $x = y$.

   (f) False. For $y > 0$, there are two values of x for which $y = x^2$, not one. For $y < 0$, there are zero values of x for which $y = x^2$.

4. Let x be a real number. For each proposition, write the contrapositive. Then prove the proposition by contraposition.

   (a) If $x^2 + 2x < 0$, then $x < 0$.
   Contrapositive: If $x \geq 0$, then $x^2 + 2x \geq 0$.
   Assume that $x \geq 0$. The we have that $2x \geq 0$. Additionally, $x^2 \geq 0$ for any real x. Summing these two inequalities gives $x^2 + 2x \geq 0$.

   (b) If $x(x - 4) > -3$, then $x < 1$ or $x > 3$.
   Contrapositive: If $x \geq 1$ and $x \leq 3$, then $x(x - 4) \leq -3$.
   Assume that $1 \leq x \leq 3$. Then $x - 1 \geq 0$ and $x - 3 \leq 0$. Since $x - 1$ is nonnegative, we can multiply both sides of the other inequality by $x - 1$ to get

$$(x - 3)(x - 1) \leq 0 \cdot (x - 1) = 0.$$

   Rearranging terms of this inequality gives $x(x - 4) \leq -3$.

5. Let a and b be positive integers. Prove each proposition by contradiction.

(a) If a divides b, then $a \leq b$.

Assume that a divides b and that $a > b$. By the definition of "divides", we have that b/a is an integer. Since a and b are positive, b/a must be positive, so $b/a \geq 1$. However, dividing both sides of the inequality $a > b$ by a gives $1 > b/a$. This is a contradiction.

(b) Either a and b are odd, or ab is even.

Assume that a or b are even and that ab is odd. First consider the case that a is even, so $a = 2k$ for some integer k. Then $ab = 2kb$, which is even. This is a contradiction. Otherwise b must be even, so $b = 2\ell$ for some integer ℓ. Then $ab = a2\ell$, which is even. This is also a contradiction. (You could also use "without loss of generality" to reduce cases here.)

(c) If $a < b$ and $ab < 4$, then $a = 1$.

Assume that $a < b$ and $ab < 4$ but $a \neq 1$. Since a is a positive integer, this implies that $a \geq 2$ and since $b > a$, we must have $b \geq 3$. Multiplying these inequalities gives $ab \geq 6$, which is a contradiction.

6. For x a real number, $\lfloor x \rfloor$ denotes the "floor" of x, which is the largest integer less than or equal to x. Prove using cases that for all integers k, the value of $\lfloor k^2/2 \rfloor$ is even.

Consider the cases that k is even or k is odd. First assume that k is even, so $k = 2m$ for some integer m. Then

$$\lfloor k^2/2 \rfloor = \lfloor (2m)^2/2 \rfloor = \lfloor 2m^2 \rfloor = 2m^2$$

which is even.

Then assume that k is odd, so $k = 2m + 1$ for some integer m. Then

$$\lfloor k^2/2 \rfloor = \lfloor (2m+1)^2/2 \rfloor = \lfloor 2m^2 + 2m + 1/2 \rfloor.$$

Since $2m^2 + 2m$ is an integer and $1/2 < 1$, the floor function rounds away the $1/2$. The value of the expression is $2m^2 + 2m$, which is even.

7. Let x, y and z be three real numbers in the interval $[0, 1]$. Prove that there exists a pair of two of the three numbers that are at distance $\leq 1/2$ apart. [Hint: You can assume without loss of generality that $x \leq y \leq z$. Why is it sufficient to only consider this case?]

Without loss of generality we can assume that $0 \leq x \leq y \leq z \leq 1$ by relabelling the numbers so that they are in order. We proceed by contradiction. Assume that no pair has distance $\leq 1/2$. Then $y - x > 1/2$ and $z - y > 1/2$. Rewrite these as $y > x + 1/2$ and $z > y + 1/2$. Since $x \geq 0$, the first inequality gives $y > 1/2$. Combining this with the second inequality gives

$$z > 1/2 + 1/2 = 1.$$

But this contradicts the fact that $z \leq 1$. Therefore there must be a pair with distance $\leq 1/2$.