MATH 108 Fall 2019 - **Problem Set 4**

due October 25

1. Let $n = a_1 a_2 \cdots a_k$ with $k \geq 1$ and $a_1, a_2, \ldots, a_k$ positive integers and let p be a prime. Use Euclid's Lemma and induction on k to prove that if p divides n, then p divides a_i for some $1 \leq i \leq k$.

2. A positive integer n is called *square-free* if it is not divisible by any perfect square except for 1. Prove that n is square-free if and only if n is a product of distinct primes.

3. For positive integers x and y, the *greatest common divisor* of x and y is the largest postive integer that divides both x and y, denoted $\gcd(x, y)$. Let a, b, c be postive integers.

   (a) Prove that $a/\gcd(a, b)$ and $b/\gcd(a, b)$ are integers that have no common factor.

   (b) For p a prime, prove that p divides a and p divides b if and only if p divides $\gcd(a, b)$.

4. For positive integers a and b with $\gcd(a, b) = d$, prove that

$$\{as + bt \mid s, t \in \mathbb{Z}\} = d\mathbb{Z}.$$

5. For each relation, list which of the following properties it has: symmetric, antisymmetric, transitive, reflexive, irreflexive.

   (a) $\leq$ on $\mathbb{Z}$.

   (b) $\neq$ on $\mathbb{Z}$.

   (c) $\subseteq$ on $\mathcal{P}(\mathbb{Z})$.

   (d) "is the child of" on people.

   (e) $\{(1, 5), (5, 1), (1, 1)\}$ on $A = \{1, 2, 3, 4, 5\}$.

   (f) $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x + y = 10\}$ on $\mathbb{Z}$.

6. Let $A = \{1, 2, 3, 4, 5\}$ and let $\sim$ be the relation on $\mathcal{P}(A)$ defined by $S \sim T$ if $|S| = |T|$.

   (a) Prove that $\sim$ is an equivalence relation.

   (b) How many equivalence classes does $\sim$ have and how many elements are in each class?

7. Let $\sim$ be a relation on set A with the property that for all $a \in A$, there exists $b \in A$ such that $a \sim b$. Prove that if $\sim$ is transitive and symmetric, then $\sim$ is reflexive.