MATH 108 Fall 2019 - **Problem Set 5 solutions**

due November 4

1. Let $\sim$ be the relation on $\mathbb{R}$ defined by $x \sim y$ if and only if $x - y \in \mathbb{Z}$.

   (a) Prove that $\sim$ is an equivalence relation.

   Reflexivity: For $x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Z}$ so $x \sim x$.

   Symmetry: For $x, y \in \mathbb{R}$, if $x \sim y$ then $x - y$ is an integer. Therefore $-(x-y) = y - x$ is also an integer, so $y \sim x$.

   Transitivity: For $x, y, z \in \mathbb{R}$, if $x \sim y$ and $y \sim z$ then $x - y$ and $y - z$ are integers. The sum of two integers is an integer, so $(x-y) + (y-z) = x - z$ is an integer. Therefore $x \sim z$.

   (b) Prove for all real numbers x, y, z, w that if $\overline{x} = \overline{z}$ and $\overline{y} = \overline{w}$ then $\overline{x+y} = \overline{z+w}$.

   Suppose that $\overline{x} = \overline{z}$ and $\overline{y} = \overline{w}$, so $x - z$ and $y - w$ are integers. The sum of two integers is an integer, so $(x-z)+(y-w) = (x+y)-(z+w)$ is an integer. Therefore $x + y \sim z + w$, meaning $\overline{x+y} = \overline{z+w}$.

2. Using modular arithmetic, prove that for all postive integers n,

   (a) $10^n - 1$ is divisible by 3.

   Since $10 \equiv 1 \pmod 3$, then

   $$10^n - 1 \equiv 1^n - 1 \equiv 0 \pmod 3$$

   for all positive integers n.

   (b) $n^4 + 2n^3 - n^2 - 2n$ is divisible by 4.

   There are four cases, depending on the equivalence class of n.

   If $n \equiv 1 \pmod 4$ then

   $$n^4 + 2n^3 - n^2 - 2n \equiv 1^4 + 2 \cdot 1^3 - 1^2 - 2 \cdot 1 \equiv 0 \pmod 4.$$

   If $n \equiv 2 \pmod 4$ then

   $$n^4 + 2n^3 - n^2 - 2n \equiv 2^4 + 2 \cdot 2^3 - 2^2 - 2 \cdot 2 \equiv 24 \equiv 0 \pmod 4.$$

   If $n \equiv 3 \pmod 4$ then

   $$n^4 + 2n^3 - n^2 - 2n \equiv 3^4 + 2 \cdot 3^3 - 3^2 - 2 \cdot 3 \equiv 120 \equiv 0 \pmod 4.$$

   If $n \equiv 0 \pmod 4$ then

   $$n^4 + 2n^3 - n^2 - 2n \equiv 0^4 + 2 \cdot 0^3 - 0^2 - 2 \cdot 0 \equiv 0 \pmod 4.$$

(c) $1^n + 2^n + 3^n + 4^n$ is a multiple of 5 or one less than a multiple of 5.

We proceed by induction on n, with base cases $1, 2, 3, 4$. For $n = 1$, we have

$$1^1 + 2^1 + 3^1 + 4^1 \equiv 10 \equiv 0 \pmod 5.$$

For $n = 2$, we have

$$1^2 + 2^2 + 3^2 + 4^2 \equiv 30 \equiv 0 \pmod 5.$$

For $n = 3$, we have

$$1^3 + 2^3 + 3^3 + 4^3 \equiv 100 \equiv 0 \pmod 5.$$

For $n = 4$, we have

$$1^4 + 2^4 + 3^4 + 4^4 \equiv 354 \equiv 4 \pmod 5.$$

For $n > 4$, assume that $1^{n-4} + 2^{n-4} + 3^{n-4} + 4^{n-4}$ is a multiple of 5 or one less than a multiple of 5. Note that

$$1^4 \equiv 2^4 \equiv 3^4 \equiv 4^4 \equiv 1 \pmod 5.$$

Therefore,

$$\begin{aligned} 1^n + 2^n + 3^n + 4^n &\equiv 1^4 \cdot 1^{n-4} + 2^4 \cdot 2^{n-4} + 3^4 \cdot 3^{n-4} + 4^4 \cdot 4^{n-4} \\ &\equiv 1^{n-4} + 2^{n-4} + 3^{n-4} + 4^{n-4} \pmod 5. \end{aligned}$$

So $1^n + 2^n + 3^n + 4^n$ is also a multiple of 5 or one less than a multiple of 5.

3. The "Cancellation Law" for $\mathbb{Z}/m\mathbb{Z}$ is the statement: For all $x, y, z \in \mathbb{Z}$, if $xy \equiv xz \pmod m$ and $x \not\equiv 0 \pmod m$ then $y \equiv z \pmod m$.

   (a) Prove that if m is prime then the Cancellation Law for $\mathbb{Z}/m\mathbb{Z}$ is true.

   Suppose that m is prime, that $xy \equiv xz \pmod m$ and that $x \not\equiv 0 \pmod m$. Then $x(y - z)$ is divisible by m and x is not divisible by m. Since m is prime, by Euclid's Lemma $y - z$ must be divisible by m. Therefore $y \equiv z \pmod m$.

   (b) Prove that if m is composite then the Cancellation Law for $\mathbb{Z}/m\mathbb{Z}$ is false.

   Supposing m is composite, we produce a counterexample to the Cancellation Law. Since m is composite, $m = xy$ for some x and y that are both not divisible by m. So $xy \equiv 0 \pmod m$ but $x \not\equiv 0 \pmod m$ and $y \not\equiv 0 \pmod m$. Let $z = 0$. Then $xy \equiv xz \pmod m$ since they are both conguent to zero. However $y \not\equiv z \pmod m$ since $y - z = y$ is not divisible by m.

4. Let $\preceq$ be the relation on $\mathbb{Z}^2$ defined by $(a, b) \preceq (c, d)$ if and only if $a \le c$ and $b \le d$.

(a) Prove that $\preceq$ is a partial order.

Reflexivity: For $(a, b) \in \mathbb{Z}^2$, $a \leq a$ and $b \leq b$ so $(a, b) \preceq (a, b)$.

Antisymmetry: For $(a, b), (c, d) \in \mathbb{Z}^2$ with $(a, b) \neq (c, d)$, either $a \neq c$ or $b \neq d$. If $a \neq c$ then either $a \not\leq c$ or $c \not\leq a$, so either $(a, b) \not\preceq (c, d)$ or $(c, d) \not\preceq (a, b)$. Similarly if $b \neq d$ then either $b \not\leq d$ or $d \not\leq b$, so either $(a, b) \not\preceq (c, d)$ or $(c, d) \not\preceq (a, b)$.

Transitivity: For $(a, b), (c, d), (e, f) \in \mathbb{Z}^2$ if $(a, b) \preceq (c, d)$ and $(c, d) \preceq (e, f)$ then $a \leq c \leq e$ and $b \leq d \leq f$ so $(a, b) \preceq (e, f)$.

(b) Find the greatest lower bound of $\{(1, 5), (3, 3)\}$.

The greatest lower bound is $(1, 3)$. Since $(1, 3) \preceq (1, 5)$ and $(1, 3) \preceq (3, 3)$, it is a lower bound. Any other lower bound $(a, b) \in \mathbb{Z}^2$ has $(a, b) \preceq (1, 5)$ and $(a, b) \preceq (3, 3)$, so $a \leq 1$ and $a \leq 3$, and $b \leq 5$ and $b \leq 3$. Therefore $a \leq 1$ and $b \leq 3$, so $(a, b) \preceq (1, 3)$. Thus any other lower bound is less than $(1, 3)$.

(c) Is $\preceq$ a total order? Justify your answer.

No. For example $(1, 5) \not\preceq (3, 3)$ and $(3, 3) \not\preceq (1, 5)$ so $(1, 5)$ and $(3, 3)$ are incomparable.

5. Let A be the set of divisors of 36, $A = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$. Draw the Hasse diagram for the poset $(A, |)$.