MATH 108 Fall 2019 - **Problem Set 7**

due November 15

1. For each function f, determine if it is surjective. If yes, find a *right-inverse* of f, which is a function g such that $f \circ g$ is the identity.

   (a) $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (x, x)$.
   Not surjective. For example $(1, 0)$ is not in the image of f.

   (b) $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x + y$.
   Surjective. Let $g : \mathbb{R} \to \mathbb{R}^2$ be the function defined by $g(x) = (x, 0)$.

   (c) $f : \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ defined by $f(x) = \overline{x}$.
   Surjective. Let $g : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}$ be the function defined by $g(\overline{0}) = 0$, $g(\overline{1}) = 1$, $g(\overline{2}) = 2$, $g(\overline{3}) = 3$.

   (d) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$.
   Not surjective. For example -1 is not in the image of f.

   (e) $f : \mathbb{Z} \to \{0\}$ defined by $f(x) = 0$.
   Surjective. Let $g : \{0\} \to \mathbb{Z}$ be the function defined by $g(0) = 0$.

2. Let $f : A \to B$ and $g : B \to C$.

   (a) Prove that if $g \circ f$ is surjective then g is surjective.
   Since $g \circ f$ is surjective, for any $z \in C$, there exists $x \in A$ such that $g \circ f(x) = z$. Let $y = f(x)$. Then $g(y) = z$, so g is surjective.

   (b) Give an example of f and g where $g \circ f$ is surjective but f is not surjective.
   Let $A = C = \{1\}$ and $B = \{1, 2\}$. Define $f : A \to B$ by $f(1) = 1$ and $g : B \to C$ by $g(1) = g(2) = 1$. Then $g \circ f$ is the identity function on $\{1\}$, which is surjective, but f is not surjective.

3. Prove that each function is a bijection. Give the inverse.

   (a) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x + 1$.
   For each $y \in \mathbb{Z}$, $x = y - 1$ is the unique integer such that $f(x) = y$. Therefore f is bijective, and the inverse $f^{-1} : \mathbb{Z} \to \mathbb{Z}$ is defined by $f^{-1}(y) = y - 1$.

   (b) $f : (2, \infty) \to (-\infty, -1)$ defined by $f(x) = \dfrac{-x}{x - 2}$.
   For $y \in (-\infty, -1)$, solving $f(x) = y$ for x shows that $x = 2y/(y + 1)$ is the unique real number that could map to y. Therefore there is at most one $x \in (2, \infty)$ with $f(x) = y$, proving f is injective.
   To see that f is surjective, we need to check that for each $y \in (-\infty, -1)$, the value $x = 2y/(y + 1)$ that would map to y is in the domain, $(2, \infty)$. Since $y < -1$, the

denominator $y+1$ is negative. Dividing both sides of the inequality $y < y+1$ by $y+1$ gives

$$\frac{y}{y+1} > 1.$$

Therefore $x = 2y/(y+1) > 2$, which proves that there is $x \in (2, \infty)$ with $f(x) = y$. The inverse is $f^{-1} : (-\infty, -1) \to (2, \infty)$ is defined by $f^{-1}(y) = 2y/(y+1)$.

(c) $f : \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$ defined by $f(\overline{x}) = \overline{5x-1}$.

$$\begin{aligned} f(\overline{0}) &= \overline{7}, \\ f(\overline{1}) &= \overline{4}, \\ f(\overline{2}) &= \overline{1}, \\ f(\overline{3}) &= \overline{6}, \\ f(\overline{4}) &= \overline{3}, \\ f(\overline{5}) &= \overline{0}, \\ f(\overline{6}) &= \overline{5}, \\ f(\overline{7}) &= \overline{2}. \end{aligned}$$

For each $\overline{y} \in \mathbb{Z}/8\mathbb{Z}$, there is exactly one $\overline{x} \in \mathbb{Z}/8\mathbb{Z}$ with $f(\overline{x}) = \overline{y}$. The inverse $f^{-1} : \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$ is defined by $f^{-1}(\overline{y}) = \overline{5(y+1)}$.

4. For each pair of sets, find a bijection from the first to the second.

   (a) $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$.
   Define $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$ by $f(x) = x - 1$.

   (b) $\mathbb{R}^2$ and $\mathbb{C}$.
   Define $f : \mathbb{R}^2 \to \mathbb{C}$ by $f(x, y) = x + iy$.

   (c) $\mathbb{Z}$ and $\mathbb{Z}_{>0}$.
   Define $f : \mathbb{Z} \to \mathbb{Z}_{>0}$ by

$$f(x) = \begin{cases} 2x+1 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}.$$

   (d) $\{x \in \mathbb{R} \mid -1 < x < 1\}$ and $\mathbb{R}$.
   Define $f : (-1, 1) \to \mathbb{R}$ by $f(x) = \frac{1}{x+1} + \frac{1}{x-1}$.

5. For postive integers n and m, let $[n] = \{1, 2, \ldots, n\}$ and $[m] = \{1, 2, \ldots, m\}$.

   (a) Let A be the set of all functions from $[n]$ to $[m]$. Compute $|A|$ in terms of n and m.
   For $f : [n] \to [m]$, we can choose the values $f(k)$ one at a time for each k from 1 up to n. For each k there are m choices for $f(k)$, so the total number of functions is m^n.

(b) Let B be the set of all bijective functions from $[n]$ to $[m]$. Compute $|B|$ in terms of n and m.

If $n \neq m$ then any function $f : [n] \to [m]$ can't be bijective, so there are 0 bijective functions.

If $n = m$, we again choose the values of $f(k)$ one at a time for each k from 1 up to n. When $k = 1$, there are n possible values for $f(1)$ to choose from. Once $f(1)$ is chosen, there are only $n - 1$ choices for $f(2)$ because $f(2)$ can't be equal to $f(1)$ if f is injective. Once $f(1)$ and $f(2)$ are chosen, there are $n - 2$ choices left for $f(3)$. This repeats for each k up to $k = n$ where we have only one choice left for $f(n)$. Therefore the number of possible bijective functions is

$$n \cdot (n-1) \cdot (n-2) \cdots 1 = n!.$$

(c) Let C be the set of all injective functions from $[n]$ to $[m]$. Compute $|C|$ in terms of n and m.

If $n > m$ then any function $f : [n] \to [m]$ can't be injective, so there are 0 injective functions.

If $n \leq m$, the analysis is similar to part (b). We can choose the values of $f(k)$ one at a time. The number of choices for $f(1)$ is m, for $f(2)$ is $m - 1$ and so on. The last decision is $f(n)$ which has $m - n + 1$ choices left. Therfore the number of possible injective functions is

$$m \cdot (m-1) \cdot (m-2) \cdots (m-n+1) = \frac{m!}{(m-n)!}.$$

6. Let $f_1, f_2 : A \to B$ and $g : B \to C$ and $h_1, h_2 : C \to D$.

(a) Prove that if $g \circ f_1 = g \circ f_2$ and g is injective, then $f_1 = f_2$.

Assume $g \circ f_1 = g \circ f_2$ and g is injective. Since g is injective, it has a left-inverse $k : C \to B$. Then $k \circ g \circ f_1 = k \circ g \circ f_2$. The left-hand side of the equation is equal to $I_B \circ f_1 = f_1$ and the right-hand side is equal to $I_B \circ f_2 = f_2$.

Alternate proof: Assume $g \circ f_1 = g \circ f_2$ and g is injective. For any $x \in A$, we have $g(f_1(x)) = g(f_2(x))$. Since g is injective, this implies $f_1(x) = f_2(x)$. Therefore f_1 and f_2 have the same domain, codomain and values so they are equal.

(b) Prove that if $h_1 \circ g = h_2 \circ g$ and g is surjective, then $h_1 = h_2$.

Assume $h_1 \circ g = h_2 \circ g$ and g is surjective. Since g is surjective, it has a right-inverse $k : C \to B$. Then $h_1 \circ g \circ k = h_2 \circ g \circ k$. The left-hand side of the equation is equal to $h_1 \circ I_C = h_1$ and the right-hand side is equal to $h_2 \circ I_C = h_2$.

Alternate proof: Assume $h_1 \circ g = h_2 \circ g$ and g is surjective. For any $y \in C$, there exists $x \in B$ such that $g(x) = y$. Since $h_1(g(x)) = h_2(g(x))$, we have $h_1(y) = h_2(y)$. Therefore h_1 and h_2 have the same domain, codomain and values so they are equal.