MATH 108 Fall 2019 - **Problem Set 8**

due November 22

1. Let X, Y, Z, W be sets with $|X| = |Z|$ and $|Y| = |W|$.

   (a) Cardinal addition is defined by $|X| + |Y| = |X \cup Y|$ where X and Y are disjoint. Prove that cardinal addition is well-defined, meaning that

   $$|X| + |Y| = |Z| + |W|$$

   where X and Y are disjoint and Z and W are disjoint.

   (b) Cardinal multiplication is defined by $|X| \cdot |Y| = |X \times Y|$. Prove that cardinal multiplication is well-defined, meaning that

   $$|X| \cdot |Y| = |Z| \cdot |W|.$$

   (c) Cardinal exponentiation is defined by $2^{|X|} = |\mathcal{P}(X)|$. Prove that cardinal exponentiation is well-defined, meaning that

   $$2^{|X|} = 2^{|Z|}.$$

2. Let n be a positive integer. Prove that the set of positive integer divisors of n is finite.

3. (a) Prove that $|\{x \in \mathbb{R} \mid -1 < x < 1\}| = |\mathbb{R}|$.

   (b) Prove that $|\{x \in \mathbb{R} \mid -1 \leq x \leq 1\}| = |\mathbb{R}|$.

4. Prove each of the following sets is countable.

   (a) The set of prime numbers.

   (b) $\mathbb{Z} \times \mathbb{Z}$.

   (d) The set of all finite-length binary strings, $\bigcup_{n=0}^{\infty} \{0,1\}^n$. (This is the set of all possible computer files.)

5. Prove that the set of irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$, is uncountable.

6. Use Cantor's diagonalization argument to prove that the set of all functions from $\mathbb{Z}_{>0}$ to $\mathbb{Z}_{>0}$ is uncountable.

7. Let X be an infinte set.

   (a) Prove that $|X| \geq \aleph_0$.

   (b) Prove that $|X| + 1 = |X|$.
   [Hint: First prove it for the case that X is countably infinite. Then for the general case, part (a) implies that X has a countably infinite subset Y. Use the fact that $|Y| + 1 = |Y|$.]