# MATH 108 Winter 2019 - **Problem Set 5**

due February 22

1. Using modular arithmetic, prove that for all postive integers $n$,
   (a) $10^n - 1$ is divisible by 3.
   (b) $n^4 + 2n^3 - n^2 - 2n$ is divisible by 4.
   (c) $1^n + 2^n + 3^n + 4^n$ is a multiple of 5 or one less than a multiple of 5.

2. The "Cancellation Law" for $\mathbb{Z}/m\mathbb{Z}$ is the statement: For all $x, y, z \in \mathbb{Z}$, if $xy \equiv xz \pmod{m}$ and $x \not\equiv 0 \pmod{m}$ then $y \equiv z \pmod{m}$.
   (a) Prove that if $m$ is prime then the Cancellation Law for $\mathbb{Z}/m\mathbb{Z}$ is true.
   (b) Prove that if $m$ is composite then the Cancellation Law for $\mathbb{Z}/m\mathbb{Z}$ is false.

3. Let $A$ and $B$ be subsets of $\mathbb{Z}$. In the poset $(\mathcal{P}(\mathbb{Z}), \subseteq)$, prove that the greatest lower bound of $\{A, B\}$ is $A \cap B$.

4. Let $A$ be the set of divisors of 36, $A = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$. Draw the Hasse diagram for the poset $(A, |)$.

5. For each function $f$, determine if it is injective. If yes, find a *left-inverse* of $f$, which is a function $g$ such that $g \circ f$ is the identity.
   (a) $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (x, x)$.
   (b) $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x + y$.
   (c) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x$.
   (d) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$.
   (e) $f : \mathbb{Z} \to \{0\}$ defined by $f(x) = 0$.

6. For each function $f$ in Problem 5, determine if it is surjective. If yes, find a *right-inverse* of $f$, which is a function $g$ such that $f \circ g$ is the identity.

7. Let $f : A \to B$ and $g : B \to C$.
   (a) Prove that if $g \circ f$ is injective then $f$ is injective.
   (b) Find an example of $f$ and $g$ where $g \circ f$ is injective but $g$ is not injective.

8. For each pair of sets, find a bijection from the first to the second.
   (a) $\mathbb{N}_1$ and $\mathbb{N}_0$.
   (b) $\mathbb{R}^2$ and $\mathbb{C}$.
   (c) $\mathbb{Z}$ and $\mathbb{N}_0$.
   (d) $\{x \in \mathbb{R} \mid -1 < x < 1\}$ and $\mathbb{R}$.