MATH 108 Winter 2019 - **Problem Set 7**

due March 8

1. (a) Prove (with the Axiom of Choice) that every infinite set has a countably infinite subset.
   (b) Prove that every infinite set has a proper subset with the same cardinality.
2. Prove that the set of irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$, is uncountable.
3. Let $\mathbb{N}_0^\infty$ denote the set of all infinite sequences of nonnegative integers,

   $$\mathbb{N}_0^\infty = \{(a_1, a_2, a_3, \ldots) \mid a_1, a_2, a_3, \ldots \in \mathbb{N}_0\}.$$

   Use Cantor's diagonalization argument to prove that $\mathbb{N}_0^\infty$ is uncountable.
4. Prove that the following sets have cardinality $\mathfrak{c}$.
   (a) The set of all functions from $\mathbb{N}_1$ to $\{0, 1\}$.
   (b) The closed interval $[0, 1]$.
   (c) $\mathcal{P}(\mathbb{N}_1) \times \mathcal{P}(\mathbb{N}_1)$.
5. Order the following cardinal numbers: $|(0, 1)|$, $|[0, 1]|$, $|\{0, 1\}|$, $|\{0\}|$, $|\mathcal{P}(\mathbb{R})|$, $|\mathbb{Q}|$, $|\emptyset|$, $|\mathbb{R}^2|$, $|\mathcal{P}(\mathcal{P}(\mathbb{R}))|$, $|\mathbb{R}|$, $|\mathcal{P}(\mathbb{Q})|$.
6. Determine whether each algebraic structure is a group. If no, which properties does it fail? If yes, is it abelian? Find an identity element if one exists.
   (a) $(\mathbb{N}_1, +)$.
   (b) $(\mathbb{Q}, \cdot)$.
   (c) $(\mathbb{Q} \setminus \{0\}, \cdot)$.
   (d) $(\mathbb{Z}/4\mathbb{Z}, +)$.
   (e) $(\mathbb{Z}/4\mathbb{Z} \setminus \{\overline{0}\}, \cdot)$.
   (f) (The set of functions $\{1, 2, 3\} \to \{1, 2, 3\}$, composition).
   (g) (The set of bijective functions $\{1, 2, 3\} \to \{1, 2, 3\}$, composition).
   (h) (The set of 2×2 real matrices with determininant 1, matrix multiplication).