MATH 108 Winter 2019: Intro to Abstract Math
Final topics

Chapters 1-3 (see midterm topics list)

Functions (Sec 4.1-4.4)

- Definition of a function
- Domain, codomain, image
- Identity functions, restrictions, inclusion maps
- Function composition
- Injectivity
- Surjectivity
- Left- and right-inverses
- Bijectivity
- (Two-way) inverses

Cardinality (Sec 5.1-5.5)

- Definition of $|A| = |B|$
- Finite cardinalities
- Countably infinite cardinality, $\aleph_0$ (and Hilbert's Hotel)
- Cantor's diagonalization argument
- Cardinality of the continuum, $\mathfrak{c}$
- Cantor's Theorem
- Cantor-Schröder-Bernstein Theorem (proving $|A| = |B|$ with injections)
- Axiom of Choice
- Comparability of cardinal numbers

Algebra (Sec 6.1-6.5)

- Binary operations
- Definition of a group and of an abelian group
- Common examples of groups: $(\mathbb{Z}, +)$, $(\mathbb{Q} \setminus \{0\}, \cdot)$, $(\mathbb{Z}/m\mathbb{Z}, +)$, $\mathfrak{S}_n$, symmetry groups, etc.
- Cayley tables
- Subgroups
- Generators
- Order of a group, order of an element
- Group homomorphisms
- Isomorphism of groups
- Definition of a ring and a field
- Ring homomorphisms

Practice Problems

1. Let $f : A \to C$ and $g : B \to D$ be functions and let $h : A \times B \to C \times D$ be defined by $h(a, b) = (f(a), g(b))$.

   (a) Prove that if f and g are injective then h is injective.

   (b) Prove that if f and g are surjective then h is surjective.

2. Determine if each function is injective and if it is surjective.

   (a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$.

   (b) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$.

   (c) $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x - y$.

   (d) $f : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ defined by $f(\overline{x}) = \overline{2x + 1}$.

3. Find a right-inverse for the quotient map $q : \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$ defined by $q(x) = \overline{x}$.

4. Prove that there exists a bijective function $f : \mathbb{R} \to \mathbb{R}^2$.

5. Find the cardinality of each set and prove your answer.

   (a) $\mathcal{P}(\mathbb{Z} \times \{1, 2, 3\})$.

   (b) $\mathbb{Q} \cap [0, 1]$.

   (c) The symmetric group on four elements, $\mathfrak{S}_4$.

6. Let A be the set of functions from $\mathbb{R}$ to $\mathbb{Z}$. Prove that A is uncountable.

7. Prove that if A is an infinite set and B is a countably infinite set, then $|A \cup B| = |A|$.

8. Prove for each pair of groups that they are not isomorphic.

   (a) $(\mathbb{Z}/5\mathbb{Z}, +)$ and $(\mathbb{Z}/6\mathbb{Z}, +)$

   (b) The symmetry group of a square and $(\mathbb{Z}/8\mathbb{Z}, +)$.

   (c) $(\mathbb{Z}, +)$ and $(\mathbb{Q} \setminus \{0\}, \cdot)$.

9. Prove that $(\mathcal{P}(\mathbb{Z}), \Delta)$ is a group where Δ denotes the symmetric difference operation defined as $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

10. Write the Cayley table for $(\mathbb{Z}/5\mathbb{Z}, +)$.

11. Find the order of each element of $(\mathbb{Z}/8\mathbb{Z}, +)$.

12. Prove that the set $\{(x, x) \mid x \in \mathbb{R}\}$ is a subgroup of $(\mathbb{R}^2, +)$.

13. Let $f : G \to H$ be a group homomorphism and let K be a subgroup of G. Prove that $\{f(k) \mid k \in K\}$ is a sugroup of H.

14. Let R be a ring with multiplicative indentity 1. For any $a \in R$ prove that $(-1) \cdot a = -a$.