MATH 150A Winter 2020 - **Problem Set 1**

due January 17

1. Write the operation table for the union operation $\cup$ on $\mathcal{P}(\{1,2\})$ (the set of all subsets of $\{1,2\}$).

2. Determine whether each set and binary operation is a group. If no, which properties does it fail? If yes, is it abelian? Find an identity element if one exists.

   (a) (The set of positive integers, $+$).

   (b) $(\mathbb{C} \setminus \{0\},\ \cdot)$.

   (c) $(\mathcal{P}(\{1,2\}),\ \cup)$.

   (d) (The set of functions $\mathbb{Z} \to \mathbb{Z}$, composition).

   (e) (The set of bijective functions $\mathbb{Z} \to \mathbb{Z}$, composition).

3. (2.1.2) Prove the following properties of inverses.

   (a) If an element a has a left-inverse ℓ and a right-inverse r, i.e. $\ell a = 1$ and $ar = 1$, then $\ell = r$, a is invertible and r is its inverse.

   (b) If a is invertible, its inverse is unique.

   (c) If a and b are invertible, then so is ab and its inverse is $b^{-1}a^{-1}$.

4. (2.2.2) Let S be a set with a binary operation that is associative and has an identity element. Prove that the subset consisting of the invertible elements in S is a group.

5. (2.2.3) Let x, y, z, w be elements of a group G.

   (a) Solve for y if $xyz^{-1}w = 1$.

   (b) Suppose that $xyz = 1$. Does it follow that $yzx = 1$? Does it follow that $yxz = 1$?

6. The *Klein four group* V is the group with 4 elements that can be represented by matrices

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}.$$

   (a) Find the order of each element of V.

   (b) Find all subgroups of V.

7. (2.2.4) In which of the following cases is H a subgroup of G? If not, say why not.

   (a) $G = \mathrm{GL}_n(\mathbb{C})$ and $H = \mathrm{GL}_n(\mathbb{R})$. ($\mathrm{GL}_n(K)$ denotes the multiplicative group of invertible $n \times n$ matrices with entries in K.)

   (b) $G = \mathbb{R}^\times$ and $H = \{-1, 1\}$.

(c) $G = (\mathbb{Z}, +)$ and H is the set of positive integers.

(d) $G = \mathbb{R}^\times$ and H is the set of positive reals.

(e) $G = \mathrm{GL}_2(\mathbb{R})$ and H is the of matrices $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, with $a \neq 0$.