# MATH 150A Winter 2020 - **Problem Set 1 solutions**

due January 17

1. Write the operation table for the union operation $\cup$ on $\mathcal{P}(\{1,2\})$ (the set of all subsets of $\{1,2\}$).

| $\cup$ | $\emptyset$ | $\{1\}$ | $\{2\}$ | $\{1,2\}$ |
|---|---|---|---|---|
| $\emptyset$ | $\emptyset$ | $\{1\}$ | $\{2\}$ | $\{1,2\}$ |
| $\{1\}$ | $\{1\}$ | $\{1\}$ | $\{1,2\}$ | $\{1,2\}$ |
| $\{2\}$ | $\{2\}$ | $\{1,2\}$ | $\{2\}$ | $\{1,2\}$ |
| $\{1,2\}$ | $\{1,2\}$ | $\{1,2\}$ | $\{1,2\}$ | $\{1,2\}$ |

2. Determine whether each set and binary operation is a group. If no, which properties does it fail? If yes, is it abelian? Find an identity element if one exists.

   (a) (The set of positive integers, $+$).

   Not a group. It doesn't have an identity element or inverses.

   (b) $(\mathbb{C} \setminus \{0\}, \cdot)$.

   Group. 1 is an identity element.

   (c) $(\mathcal{P}(\{1,2\}), \cup)$.

   Not a group. The non-empty sets don't have inverses. $\emptyset$ is an identity element.

   (d) (The set of functions $\mathbb{Z} \to \mathbb{Z}$, composition).

   Not a group. The functions that are not bijective don't have inverses. The identity function is an identity element.

   (e) (The set of bijective functions $\mathbb{Z} \to \mathbb{Z}$, composition).

   Group. The identity function is an identity element.

3. (2.1.2) Prove the following properties of inverses.

   (a) If an element $a$ has a left-inverse $\ell$ and a right-inverse $r$, i.e. $\ell a = 1$ and $ar = 1$, then $\ell = r$, $a$ is invertible and $r$ is its inverse.

$$\ell = \ell(ar) = (\ell a)r = r.$$

   Since $\ell = r$, $r$ is both a left and right-inverse of $a$, so $a$ is invertible.

   (b) If $a$ is invertible, its inverse is unique.

   Let $b_1$ and $b_2$ be inverses of $a$. Then

$$b_1 = b_1(ab_2) = (b_1 a)b_2 = b_2.$$

   Therefore the inverse of $a$ is unique.

(c) If $a$ and $b$ are invertible, then so is $ab$ and its inverse is $b^{-1}a^{-1}$.

$$(b^{-1}a^{-1})(ab) = b^{-1}a^{-1}ab = b^{-1}b = 1.$$
$$(ab)(b^{-1}a^{-1}) = abb^{-1}a^{-1} = aa^{-1} = 1.$$

Therefore $b^{-1}a^{-1}$ is the inverse of $ab$.

4. (2.2.2) Let $S$ be a set with a binary operation that is associative and has an identity element. Prove that the subset consisting of the invertible elements in $S$ is a group.

   Let $*$ be the operation on $S$ and let

   $$T = \{x \in S \mid \exists y \in S \text{ such that } y \text{ is an inverse of } x\}.$$

   To prove that $T$ is a group, we need to show that $T$ is closed under the operation $*$, that $*$ is associative on $T$, that the identity element of $*$ is in $T$, and that for each $x \in T$ the inverse of $x$ is also in $T$.

   By Problem 3c, if $a$ and $b$ are invertible, then so is $a * b$. Therefore if $a, b \in T$, then $a * b \in T$, so $T$ is closed under $*$.

   Since $*$ is associative on $S$, it is also associative on a subset $T \subseteq S$.

   The identity element $e$ of $*$ is invertible because $e$ is the inverse of $e$, so $e \in T$.

   If $x \in T$, then it has an inverse $y \in S$. Then $x$ is also an inverse of $y$, so $y \in T$. Therefore every element of $T$ has an inverse in $T$.

   $T$ satisfies all of the properties needed to be a group.

5. (2.2.3) Let $x, y, z, w$ be elements of a group $G$.

   (a) Solve for $y$ if $xyz^{-1}w = 1$.
   Multiply both sides of the equation on the left by $x^{-1}$

   $$x^{-1}xyz^{-1}w = x^{-1}1,$$
   $$yz^{-1}w = x^{-1}.$$

   Then multiply both sides of the equation on the right by $w^{-1}z$

   $$yz^{-1}ww^{-1}z = x^{-1}w^{-1}z,$$
   $$y = x^{-1}w^{-1}z.$$

   (b) Suppose that $xyz = 1$. Does it follow that $yzx = 1$? Does it follow that $yxz = 1$?
   The answer to the first question is yes. Multiply both sides of the equation on the left by $x^{-1}$ and on the right by $x$

   $$x^{-1}xyzx = x^{-1}1x$$
   $$yzx = 1.$$

   However it does not follow that $yxz = 1$. To construct a counterexample, choose $G$ to be your favorite non-abelian group and choose $x$ and $y$ to be elements that don't commute: $xy \neq yx$. Then let $z = (xy)^{-1}$. It follows that $z \neq (yx)^{-1}$ because $z$ can only have one inverse. Then $xyz = 1$, but $yxz \neq 1$.

6. The *Klein four group* $V$ is the group with 4 elements that can be represented by matrices

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}.$$

   (a) Find the order of each element of $V$.

   The identity matrix has order 1, and the other three elements have order 2.

   (b) Find all subgroups of $V$.

   - $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$
   - $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$
   - $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$
   - $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$
   - $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

7. (2.2.4) In which of the following cases is $H$ a subgroup of $G$? If not, say why not.

   (a) $G = \mathrm{GL}_n(\mathbb{C})$ and $H = \mathrm{GL}_n(\mathbb{R})$. ($\mathrm{GL}_n(K)$ denotes the multiplicative group of invertible $n \times n$ matrices with entries in $K$.)

   Subgroup.

   (b) $G = \mathbb{R}^\times$ and $H = \{-1, 1\}$.

   Subgroup.

   (c) $G = (\mathbb{Z}, +)$ and $H$ is the set of positive integers.

   Not a subgroup. $H$ does not have the identity element nor inverses.

   (d) $G = \mathbb{R}^\times$ and $H$ is the set of positive reals.

   Subgroup.

   (e) $G = \mathrm{GL}_2(\mathbb{R})$ and $H$ is the of matrices $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, with $a \neq 0$.

   Not a subgroup. $H$ is a group, but $H \not\subseteq G$ because the elements of $H$ are not invertible matrices.