MATH 150A Winter 2020 - **Problem Set 4**

due February 7

1. Let G be a group with identity element e.
   (a) Prove that G/G is trivial.
   (b) Prove that $G/\{e\}$ is isomorphic to G.

2. Let G be the additive group of $\mathbb{R}^2$ and H the subgroup
$$H = \{(x, x) \mid x \in \mathbb{R}\}.$$
   (a) Characterize all left cosets of H.
   (b) Prove that G/H is isomorphic to the additive group of $\mathbb{R}$.
   (c) Find a homomorphism $f : \mathbb{R}^2 \to \mathbb{R}$ with $\ker f = H$.

3. (2.12.4) Let $G = \mathbb{C}^\times$ and $H = \{\pm 1, \pm i\}$, the subgroup of fourth roots of unity.
   (a) Characterize all left cosets of H.
   (b) Prove or disprove that G/H isomorphic to G.

4. (2.12.1) Show that if a subgroup H of a group G is not normal, then there are left cosets aH and bH whose product is not a coset of H.

5. Let K be a normal subgroup of G and $q : G \to G/K$ be the quotient map. Let $f : G \to H$ be a homomorphism with $K \subseteq \ker(f)$. Prove that f factors through q, meaning that there exists a homomorphism $\varphi : G/K \to H$ such that $f = \varphi \circ q$.

6. (2.11.1) Let x be an element of group G with order r, and let y be an elements of group H with order s. Find the order of (x, y) in the product group $G \times H$.

7. (2.11.9) Let H and K be subroups of group G. Prove that the product set HK is a subgroup of G if and only if $HK = KH$.

8. Let G and H be groups. Prove that $G \times \{1\}$ is a normal subgroup of $G \times H$. Prove that $(G \times H)/(G \times \{1\})$ is isomorphic to H.