MATH 150A Winter 2020 - **Problem Set 4 solutions**

due February 7

1. Let G be a group with identity element e.

   (a) Prove that G/G is trivial.
   G/G is defined as the set of left cosets of G,

   $$G/G = \{aG \mid a \in G\}.$$

   However every coset aG is equal to G, so $G/G = \{G\}$ which is trivial.

   (b) Prove that $G/\{e\}$ is isomorphic to G.
   Each left coset of the trivial group has the form $a\{e\} = \{ae\} = \{a\}$, so the quotient map $q : G \to G/\{e\}$ is given by $q(a) = \{a\}$ for each $a \in G$. It follows that q is injective, and since we already know the quotient map is a surjective homomorphism, q is an isomorphism.

2. Let G be the additive group of $\mathbb{R}^2$ and H the subgroup

   $$H = \{(x, x) \mid x \in \mathbb{R}\}.$$

   (a) Characterize all left cosets of H.
   The left coset for the point (a, b) is given by

   $$(a, b)H = \{(a + x, b + x) \mid x \in \mathbb{R}\}.$$

   This is the line given by the equation $y = x + b - a$. So the left cosets are all lines of the form $y = x + c$ for all $c \in \mathbb{R}$.

   (b) Prove that G/H is isomorphic to the additive group of $\mathbb{R}$.
   Define a function $g : G/H \to \mathbb{R}$ that sends the line $y = x + c$ to c. Equivalently g can be defined by $g(\overline{(a, b)}) = b - a$. For the latter definition, we need to check that this is well-defined. Suppose $\overline{(a, b)} = \overline{(c, d)}$, so $(c, d) = (a + x, b + x)$ for some $x \in \mathbb{R}$. Then

   $$g(\overline{(c, d)}) = d - c = (b + x) - (a + x) = b - a = g(\overline{(a, b)})$$

   so the function is well-defined.
   To prove g is injective, suppose that $g(\overline{(a, b)}) = g(\overline{(c, d)})$, so $b - a = d - c$. Let $x = c - a$. Then $(c, d) = (a + x, b + x)$ so $\overline{(a, b)} = \overline{(c, d)}$.
   To prove g is surjective, note that for any $c \in \mathbb{R}$, $g(\overline{(0, c)}) = c$.
   Finally, to prove that g is a homomorphsm,

   $$g(\overline{(a, b)} + \overline{(c, d)}) = \overline{(a + c, b + d)} = (b + d) - (a + c) = (b - a) + (d - c) = g(\overline{(a, b)}) + \overline{(c, d)}.$$

(c) Find a homomorphism $f : \mathbb{R}^2 \to \mathbb{R}$ with $\ker f = H$.

Let $q : G \to G/H$ be the quotient map, which has kernel H, and $g : G/H \to \mathbb{R}$ be the isomorphism defined in part (b). Then $f = g \circ q : G \to \mathbb{R}$ is a homomorphism with $\ker f = H$. This map f is defined as $f(a, b) = b - a$.

3. (2.12.4) Let $G = \mathbb{C}^\times$ and $H = \{\pm 1, \pm i\}$, the subgroup of fourth roots of unity.

(a) Characterize all left cosets of H.

For each non-zero complex number z, the coset for z is

$$zH = \{\pm z, \pm iz\}.$$

(b) Prove or disprove that G/H isomorphic to G.

Let $f : G/H \to G$ be defined by $f(\overline{z}) = z^4$. For each $c \in \mathbb{C}^\times$, there are exactly 4 roots of the equation $z^4 = c$. If z is one of the roots, then the set of roots is $\{\pm z, \pm iz\} = zH$. Therefore there is exactly one coset in G/H mapping to each $c \in G$, so f is a bijection. To prove it is a homomorphism,

$$f(\overline{z} \cdot \overline{w}) = f(\overline{zw}) = (zw)^4 = z^4 w^4 = f(\overline{z}) f(\overline{w}).$$

4. (2.12.1) Show that if a subgroup H of a group G is not normal, then there are left cosets aH and bH whose product is not a coset of H.

Suppose H is not a normal subgroup of G, so there is $h \in H$ and $g \in G$ such that $ghg^{-1} \notin H$. Let $b = g^{-1}$ and let a be some other element of G. Since H is a subgroup, it also contains 1 and h^{-1}. The product set $aHbH$ contains both the elements $x = a1b1 = ab$ and $y = ah^{-1}b1 = ah^{-1}b$. If $aHbH$ were a coset, then x and y would satisfy $y^{-1}x \in H$. However

$$y^{-1}x = (ah^{-1}b)^{-1}ab = b^{-1}ha^{-1}ab = b^{-1}hb = ghg^{-1}$$

which is not in H so x and y are in different cosets. Therefore $aHbH$ is not a coset.

5. Let K be a normal subgroup of G and $q : G \to G/K$ be the quotient map. Let $f : G \to H$ be a homomorphism with $K \subseteq \ker(f)$. Prove that f factors through q, meaning that there exists a homomorphism $\varphi : G/K \to H$ such that $f = \varphi \circ q$.

We want to define a map φ such that $f(a) = \varphi(q(a))$ for all $a \in G$. Since $q(a) = \overline{a}$, we define φ by $\varphi(\overline{a}) = f(a)$. To check that this φ is well-defined, suppose that $\overline{a} = \overline{b}$, so $b = ak$ for some $k \in K$. Then

$$\varphi(\overline{b}) = f(b) = f(ak) = f(a)f(k).$$

Because $k \in K \subseteq \ker(f)$, we have $f(k) = 1$ so

$$f(a)f(k) = f(a) \cdot 1 = f(a) = \varphi(\overline{a}).$$

To check that φ is a homomorphism,

$$\varphi(\overline{a} \cdot \overline{b}) = \varphi(\overline{ab}) = f(ab) = f(a)f(b) = \varphi(\overline{a})\varphi(\overline{b}).$$

6. (2.11.1) Let x be an element of group G with order r, and let y be an elements of group H with order s. Find the order of (x, y) in the product group $G \times H$.

   The order of (x, y) is the smallest positive exponent n such that $(x, y)^n = (1, 1)$. Since $(x, y)^n = (x^n, y^n)$, this is the smallest positive integer n that is a multiple of both r and s. Therefore $n = \text{lcm}(r, s)$.

7. (2.11.9) Let H and K be subroups of group G. Prove that the product set HK is a subgroup of G if and only if $HK = KH$.

   Suppose $HK = KH$. For h_1k_1 and h_2k_2 in HK, we want to show that $(h_1k_1)(h_2k_2)^{-1}$ is also in HK.

   $$(h_1k_1)(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1}.$$

   The factor $k_1k_2^{-1}h_2^{-1}$ is in $KH = HK$ so it can be expressed as $h'k'$ for some $h' \in H$ and $k' \in K$.

   $$h_1k_1k_2^{-1}h_2^{-1} = h_1h'k'$$

   and $h_1h' \in H$ and $k' \in K$ so the product is in HK.

   Suppose that HK is a subgroup. Each $kh \in KH$ is the inverse of $h^{-1}k^{-1} \in HK$. Since HK is a group, that implies $kh \in HK$. Therefore $KH \subseteq HK$.

   For each $hk \in HK$, $(hk)^{-1} = k^{-1}h^{-1} \in HK$, so $k^{-1}h^{-1} = h'k'$ for some $h' \in H$ and $k' \in K$. Then $hk = (h'k')^{-1} = (k')^{-1}(h')^{-1} \in KH$. Therefore $HK \subseteq KH$.

8. Let G and H be groups. Prove that $G \times \{1\}$ is a normal subgroup of $G \times H$. Prove that $(G \times H)/(G \times \{1\})$ is isomorphic to H.

   Let $(g, 1) \in G \times \{1\}$ and $(a, b) \in G \times H$. Then

   $$(a, b)(g, 1)(a, b)^{-1} = (aga^{-1}, b1b^{-1}) = (aga^{-1}, 1) \in G \times \{1\}$$

   so $G \times \{1\}$ is normal.

   In the quotient $(G \times H)/(G \times \{1\})$, the equivalence classes are

   $$\overline{(g, h)} = \{(gk, h) \mid k \in G\} = G \times \{h\},$$

   so there is one equivalence class for each element $h \in H$. The equivalence class $G \times \{h\}$ can be written as $\overline{(1, h)}$. Let $f : H \to (G \times H)/(G \times \{1\})$ be the map $f(h) = \overline{(1, h)}$. This is clearly a bijection. To prove it is a homomorphism,

   $$f(a)f(b) = \overline{(1, a)(1, b)} = \overline{(1, ab)} = f(ab).$$