MATH 150A Winter 2020 - **Problem Set 5**

due February 19

1. Let C_n denote the cyclic group of order n.

   (a) For which pairs of positive integers n and m is $C_n \times C_m$ cyclic?

   (b) Prove that $\mathbb{Z} \times \mathbb{Z}$ is not cyclic.

2. Let G and H be groups and $\varphi : H \to \mathrm{Aut}(G)$ a homomorphism. The *semidirect product group*, $G \rtimes_\varphi H$, is defined as the set $G \times H$ with operation

$$(g_1, h_1)(g_2, h_2) = (g_1\varphi(h_1)(g_2), h_1h_2).$$

   (a) Prove that $G \rtimes_\varphi H$ is a group.

   (b) Prove that $G \times \{1\}$ is a normal subgroup of $G \rtimes_\varphi H$.

3. Let D_n denote the dihedral group for a regular n-gon with $n \geq 3$. Show that D_n has a semidirect product structure,

$$D_n \cong C_n \rtimes_\varphi C_2.$$

   What is $\varphi : C_2 \to \mathrm{Aut}(C_n)$ in this case?

4. (7.1.2) Let H be a subgroup of group G. Describe the orbits of the H-action on G by left multiplication.

5. $\mathrm{O}(n)$ denotes the *orthogonal group*, the subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of all real orthogonal $n \times n$ matrices. These are the rotations and reflections of $\mathbb{R}^n$ that fix the origin. Find the orbits of the $\mathrm{O}(2)$-action on $\mathbb{R}^2$. For a point $(x, y) \in \mathbb{R}^2$ what is its stabilizer?