MATH 108 Fall 2019 - **Problem Set 2 solutions**

due October 11

1. Let x and y be real numbers.

   (a) Prove for all x and y that if $x + y$ is irrational then x is irrational or y is irrational.
   Proceed by contraposition, so assume that x and y are both rational, so $x = a/b$ and $y = c/d$ for some integers a, b, c, d. The sum of two rational numbers is rational:

   $$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$$

   since $ad + cb$ and bd are integers.

   (b) Prove for all x that there exists y such that $x + y$ is rational.
   Let x be a real number. Then let $y = -x$, so $x + y = 0$ which is rational.

2. For all integers x, prove that x is divisible by 6 if and only if x is divisible by 2 and by 3.

   Assume that x is divisible by 6 so $x = 6k$ for some integr k. Then $x = 2(3k)$ so x is divisible by 2, and $x = 3(2k)$ so x is divisible by 3.

   Assume that x is divisible by 2 and 3. Since x is divisible by 3, $x = 3k$ for some integer k. Since x is even, either 3 is even or k is even. But 3 is not even, so k is even. Therefore $k = 2\ell$ for some integer ℓ. Then $x = 3k = 6\ell$, so it is divisible by 6.

3. (a) Prove that there exist integers m and n such that $3m + 4n = 1$.
   Let $m = -1$ and $n = 1$. Then $3m + 4n = 3(-1) + 4(1) = 1$.

   (b) Prove that there does not exist integers m and n such that $3m + 6n = 1$.
   Since $3m + 6n = 3(m + 2n)$ and $m + 2n$ is an integer, $3m + 6n$ is divisible by 3 for all integers m and n. On the other hand 1 is not divisible by 3. Therefore $3m + 6n$ cannot be equal to 1.

4. Let $A = \{1, 2\}$ and $B = \{1, 4, 5\}$.

   (a) Find $A \cup B$.
   $A \cup B = \{1, 2, 4, 5\}$.

   (b) Find $A \cap B$.
   $A \cap B = \{1\}$.

   (c) Find $A \setminus B$.
   $A \setminus B = \{2\}$.

   (d) Find $A \times B$.
   $A \times B = \{(1, 1), (1, 4), (1, 5), (2, 1), (2, 4), (2, 5)\}$.

(e) Find $\mathcal{P}(A)$.
$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.

5. Let A, B, C, D be sets. Prove the following propositions.

(a) $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.
Suppose $x \in (A \cup B) \cap C$. Then $x \in A \cup B$ and $x \in C$, which means that either $x \in A$ and $x \in C$, or $x \in B$ and $x \in C$. In the former case, $x \in A$. In the latter case, $x \in B$ and $x \in C$ so $x \in B \cap C$. So in either case, $x \in A \cup (B \cap C)$.

(b) $(A \setminus B) \cap (A \setminus C) = A \setminus (B \cup C)$.
Suppose $x \in (A \setminus B) \cap (A \setminus C)$, which means that $x \in A$ and $x \notin B$, and that $x \in A$ and $x \notin C$. Since $x \notin B$ and $x \notin C$, we have $x \notin B \cup C$. Therefore $x \in A \setminus (B \cup C)$. This proves $(A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C)$.

Suppose $x \in A \setminus (B \cup C)$, which means that $x \in A$ and $x \notin B \cup C$. Since $x \notin B \cup C$, we have $x \notin B$ and $x \notin C$. Therefore $x \in A \setminus B$ and also $x \in A \setminus C$. So $(A \setminus B) \cap (A \setminus C) \supseteq A \setminus (B \cup C)$.

(c) If A and B are disjoint, then $A \cap C$ and $B \cap C$ are disjoint.
Suppose A and B are disjoint, so $A \cap B = \emptyset$. Then

$$(A \cap C) \cap (B \cap C) = A \cap B \cap C \subseteq A \cap B = \emptyset.$$

So then $(A \cap C) \cap (B \cap C) = \emptyset$, which means $A \cap C$ and $B \cap C$ are disjoint.

(d) If $C \subseteq A$ and $D \subseteq B$ then $D \setminus A \subseteq B \setminus C$.
Assume that $C \subseteq A$ and $D \subseteq B$. Suppose $x \in D \setminus A$, so $x \in D$ and $x \notin A$. We have from $D \subseteq B$ that if $x \in D$ then $x \in B$, so we can conclude that $x \in B$. We have from $C \subseteq A$ that if $x \in C$ then $x \in A$. The contrapositive of this statement combined with $x \notin A$ implies $x \notin C$. Therefore $x \in B \setminus C$.

6. Let A be the set of positive integers that are not perfect squares. Let P be the set of prime numbers. Prove that $P \subseteq A$.

We proceed by contradiction. Assume that $P \not\subseteq A$, so there exists some $p \in P$ such that $p \notin A$. Since $p \notin A$, it is a square, so $p = n^2$ for some integer n. Since p is a prime, $p > 1$, so $n > 1$ as well. However $p = n \cdot n$ with $n > 1$ implies that p is composite. This contradicts $p \in P$.

7. Let S be a set of 4 distinct integers. Prove that there exists a pair of distinct elements $x, y \in S$ such that $x - y$ is divisible by 3.

When performing integer division by 3 on an integer, the remainder can be 0, 1 or 2. Since S has 4 elements, by pigeon-hole principle, there must be two elements $x, y \in S$ that have the same remainder, r. So $x = 3n + r$ and $y = 3m + r$ for some integers n, m. Then

$$x - y = (3n + r) - (3m + r) = 3(n - m).$$

which is divisible by 3.