MATH 108 Fall 2019 - **Problem Set 3 solutions**

due October 18

1. For each postive integer k, let $A_k = \{x \in \mathbb{R} \mid 0 < x < 1/k\}$. Prove that

$$\bigcap_{k=1}^{\infty} A_k = \emptyset.$$

Let $S = \bigcap_{k=1}^{\infty} A_k$. To prove that S is empty, it is sufficient to show that for all $x \in \mathbb{R}$, $x \notin S$. If $x \leq 0$ then $x \notin A_k$ for any value of k, so $x \notin S$. Suppose that $x > 0$. Then there exists a positive integer k such that $k > 1/x$. Then $x > 1/k$ so $x \notin A_k$. Therefore $x \notin S$.

2. Using induction, prove that for all positive integers n,

(a) $n^3 - n$ is divisible by 3.

The base case is $n = 1$. $1^3 - 1 = 0$ which is divisible by 3.

Assume for some $n \geq 1$ that $n^3 - n$ is divisible by 3. Then

$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1$$

$$= (n^3 - n) + 3n^2 + 3n.$$

Since $3n^2$ and $3n$ are multiples of 3 and $n^3 - n$ is divisible by 3, the sum is also divisible by 3.

(b) $8^n - 1$ is divisible by 7.

The base case is $n = 1$. $8^1 - 1 = 7$ which is divisible by 7.

Assume for some $n \geq 1$ that $8^n - 1$ is divisible by 7. Then

$$8^{n+1} - 1 = 8 \cdot 8^n - 1$$

$$= 8 \cdot 8^n - 8 + 7 = 8(8^n - 1) + 7.$$

Since $8(8^n - 1)$ is divisible by $8^n - 1$, it is divisible by 7. Clearly 7 is also divisible by 7, so the sum is divisible by 7.

(c) $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

The base case is $n = 1$. $\sum_{k=1}^{1} k^3 = 1^3 = 1$. On the other side, $\frac{1^2(1+1)^2}{4} = 1$, so the equality holds.

Assume for some $n \geq 1$ that $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$. Then

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3.$$

By the induction hypothesis,

$$= \frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{(n^4+2n^3+n^2)+(4n^3+12n^2+12n+4)}{4}$$
$$= \frac{n^4+6n^3+13n^2+12n+4}{4}.$$

On the other side,

$$\frac{(n+1)^2(n+2)^2}{4} = \frac{(n^2+2n+1)(n^2+4n+4)}{4}$$
$$= \frac{n^4+6n^3+13n^2+12n+4}{4}$$

so the equality holds for $n+1$.

(d) $n! = 1 + \sum_{k=1}^{n-1} k \cdot k!$.
The base case is $n = 1$. $1! = 1$. On the other side, $1 + \sum_{k=1}^{0} k \cdot k! = 1 + 0$, so the equality holds.
Assume for some $n \geq 1$ that $n! = 1 + \sum_{k=1}^{n-1} k \cdot k!$. Then

$$1 + \sum_{k=1}^{n} k \cdot k! = 1 + \sum_{k=1}^{n-1} k \cdot k! + n \cdot n!$$

By the induction hypothesis,

$$= n! + n \cdot n! = (n+1)n! = (n+1)!.$$

3. In American football, a team can score seven points for a touchdown, and three points for a field goal (ignore safeties, two-point conversions, etc). Prove that every integer score larger than 11 is possible.

The base cases are $n = 12, 13, 14$. 12 points can be obtained from 4 field goals. 13 points can be obtained from a touchdown and two field goals. 14 points can be obtained from two touchdowns.

For $n \geq 15$ assume that $n - 3$ points is possible, so $n - 3 = 3a + 7b$ for some nonnegative integers a and b. then

$$n = 3(a+1) + 7b$$

so n points can be obtained by $a+1$ field goals and b touchdowns.

4. Let P be the set of prime numbers. Prove that

$$\bigcup_{p \in P} p\mathbb{Z} = \mathbb{Z} \setminus \{-1, 1\}.$$

Assume that $x \in \bigcup_{p \in P} p\mathbb{Z}$, so $x \in p\mathbb{Z}$ for some prime p. Since $p\mathbb{Z} \subseteq \mathbb{Z}$, x is an integer. However 1 and -1 are not divisible by any prime p, since $p \geq 2$, so $x \neq 1$ and $x \neq -1$. Therefore $x \in \mathbb{Z} \setminus \{-1, 1\}$. This proves $\bigcup_{p \in P} p\mathbb{Z} \subseteq \mathbb{Z} \setminus \{-1, 1\}$.

Assume that $x \in \mathbb{Z}\backslash\{-1,1\}$. We consider three cases. First assume x is positive, so $x \geq 2$. By the theorem showed in class, x is divisible by a prime number p, so $x \in p\mathbb{Z} \subseteq \bigcup_{p\in P} p\mathbb{Z}$. Next assume x is negative, so $x \leq -2$. Then $-x$ is divisible by a prime p by the theorem. So $-x = kp$ for some integer k. Then $x = (-k)p$ so $x \in p\mathbb{Z} \subseteq \bigcup_{p\in P} p\mathbb{Z}$. Finally assume $x = 0$. Then for any prime p, $x = 0 \cdot p$, so $x \in p\mathbb{Z} \subseteq \bigcup_{p\in P} p\mathbb{Z}$. This proves $\bigcup_{p\in P} p\mathbb{Z} \supseteq \mathbb{Z} \setminus \{-1,1\}$.

5. Use the Well-Ordering Principle of the natural numbers to prove that every positive rational number x can be expressed as a fraction $x = a/b$ where a and b are postive integers with no common factor.

   Let
$$S = \{a \in \mathbb{Z}_{>0} \mid \exists b \in \mathbb{Z} \text{ s.t. } x = a/b\}.$$

   Since x is a rational number, it can be expressed as a fraction of integers $x = c/d$. Since x is positive, either c is positive, so $c \in S$, or else c is negative and $x = (-c)/(-d)$, so $-c \in S$. Therefore S is not empty. By the Well-Ordering Principle, S has a smallest element, a.

   Suppose that $x = a/b$ and that a and b have a common factor $d > 1$. Then $a = dk$ and $b = d\ell$ for some integers k and ℓ. Note then that $0 < k < a$. We have $x = (dk)/(d\ell) = k/\ell$ with $k > 0$ so then $k \in S$. But this contradicts the fact that a was the smallest element of S. Therefore it must be that a and b have no common factor. Finally, since x and a are positive, b must also be positive.

6. The Fibonacci sequence is an infinite sequence of integers $(f_0, f_1, f_2, f_3, \ldots)$ defined as follows. The first two numbers are $f_0 = 0$ and $f_1 = 1$. For all $n \geq 2$, define f_n to be the sum of the previous two numbers,
$$f_n = f_{n-1} + f_{n-2}.$$

   Use induction to prove that for all nonnegative integers n,
$$f_n = \frac{\varphi^n - \psi^n}{\varphi - \psi},$$
   where $\varphi = (1+\sqrt{5})/2$ and $\psi = (1-\sqrt{5})/2$.

   The base cases are $n = 0, 1$. For $n = 0$,
$$\frac{\varphi^0 - \psi^0}{\varphi - \psi} = \frac{0}{\varphi - \psi} = 0 = f_0.$$

   For $n = 1$,
$$\frac{\varphi^1 - \psi^1}{\varphi - \psi} = \frac{\varphi - \psi}{\varphi - \psi} = 1 = f_1.$$

   Assume for some $n \geq 2$ that $f_k = \frac{\varphi^k - \psi^k}{\varphi - \psi}$ for all $0 \leq k < n$. Then
$$f_n = f_{n-1} + f_{n-2}$$

and by the induction hypothesis

$$= \frac{\varphi^{n-1} - \psi^{n-1}}{\varphi - \psi} + \frac{\varphi^{n-2} - \psi^{n-2}}{\varphi - \psi} = \frac{(\varphi^{n-1} + \varphi^{n-2}) - (\psi^{n-1} + \psi^{n-2})}{\varphi - \psi}.$$

We have

$$\varphi^{n-1} + \varphi^{n-2} = (\varphi + 1)\varphi^{n-2} = \frac{3 + \sqrt{5}}{2}\varphi^{n-2}$$

$$= \left(\frac{1 + \sqrt{5}}{2}\right)^2 \varphi^{n-2} = \varphi^2 \cdot \varphi^{n-2} = \varphi^n.$$

Similarly

$$\psi^{n-1} + \psi^{n-2} = (\psi + 1)\psi^{n-2} = \frac{3 - \sqrt{5}}{2}\psi^{n-2}$$

$$= \left(\frac{1 - \sqrt{5}}{2}\right)^2 \varphi^{n-2} = \psi^2 \cdot \psi^{n-2} = \psi^n.$$

Therefore

$$f_n = \frac{\varphi^n - \psi^n}{\varphi - \psi}.$$

7. Nim is a two-player game involving piles of coins. The players alternate taking turns, and on each turn the player chooses a nonempty pile and chooses a positive number of coins to remove from that pile. This continues until there are no coins left. In this version of Nim, whoever takes the last coin loses, and the game starts with two piles, each with n coins. Prove by induction that for all $n \geq 2$, the second player has a winning strategy, i.e. they can always win no matter what the first player does.

   We proceed by strong induction. Assume for some $n \geq 2$ that player 2 has a winning strategy when the two piles start with k coins for all $2 \leq k < n$.

   Let the starting number of coins in each pile be n. Player 1 must remove m coins for some $0 < m \leq n$ from one of the piles. Call the pile that player 1 chooses pile 1, and the other pile 2. If $m = n$, then pile 1 is empty. Player 2 should remove $n - 1$ coins from pile 2, leaving only one coin. Now player 1 must remove the last coin, and loses. If $m = n-1$, then player 2 should remove all n coins from pile 2, leaving only one coin. Then player 1 loses. If $m < n-1$, then player 2 should remove m coins from pile 2. Now both piles have $n - m$ coins with $n - m \geq 2$ and it is player 1's turn again. By the induction hypothesis, player 2 has a winning strategy from this position. Therefore player 2 can always win, no matter what player 1 does.