MATH 150A Winter 2020 - **Problem Set 2**

due January 24

1. (a) Characterize the elements of $\mathbb{C}^\times$ that have order n for positive integer n.
   (b) Characterize the elements of $\mathbb{C}^\times$ that have order ∞.

2. (2.4.3) Let a and b be elements of a group G. Prove that ab and ba have the same order.

3. (2.4.10) Show by example that the product of elements of finite order in a group need not have finite order. What if the group is abelian?

4. (2.5.2) Let H and K be subgroups of group G.
   (a) Prove that the intersection $K \cap H$ is a subgroup of H.
   (b) Prove that if K is a normal subgroup of G, then $K \cap H$ is a normal subgroup of H.

5. (a) Find an injective homomorphism from the symmetric group S_3 to $\mathrm{GL}_3(\mathbb{R})$.
   (b) Let C_8 denote the cyclic group of order 8. Find an injective homomorphism from C_8 to $\mathrm{GL}_2(\mathbb{R})$.

6. (2.5.4) Let $f : \mathbb{R}^+ \to \mathbb{C}^\times$ be the map defined by $f(x) = e^{ix}$. Prove that f is a homomorphism, and determine its kernel and image.

7. Let D_5 denote the *dihedral group of the pentagon*, which is the group of order 10 consisting of the symmetries of a regular pentagon in the plane. D_5 is generated by r and s which represent a counter-clockwise rotation of the pentagon by $2\pi/5$ radians, and a reflection, respectively. Find all subgroups of D_5 and determine which subgroups are normal.

8. (a) Prove that if $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ is a group homomorphism, then $f(x) = cx$ for some constant c.
   (b) Let V and W be vector spaces over $\mathbb{Q}$ and $T : V \to W$ a function. Prove that T is a group homomorphism between $(V, +)$ and $(W, +)$ if and only if T is a linear map.
   (c) Is the property in part (a) true for $f : \mathbb{C}^+ \to \mathbb{C}^+$?
   (d) Is the property in part (a) true for $f : \mathbb{R}^+ \to \mathbb{R}^+$?