MATH 150A Winter 2020 - **Problem Set 3**

due January 31

1. Let G be a group generated by set A. Prove that if a and b commute for all $a, b \in A$, then G is abelian.

2. Prove that if group G has order 4 then G is cyclic or G is isomorphic to the Klein four group.

3. For group G, $\text{Aut}(G)$ denotes the *automorphism group* of G, whose elements are all automorphisms $G \to G$ and with composition as the operation.

   (a) Prove that $\text{Aut}(G)$ is in fact a group.

   (b) Let $\gamma : G \to \text{Aut}(G)$ be defined by $g \mapsto \varphi_g$ where $\varphi_g : G \to G$ is the map that conjugates by g, $\varphi_g(x) = gxg^{-1}$. Prove that γ is a group homomorphism.

4. (2.5.2) Find all automorphisms of

   (a) the cyclic group of order 10,

   (b) the symmetric group S_3.

5. (2.7.1) Let G be a group and define the relation $\sim$ on G by $a \sim b$ if $b = gag^{-1}$ for some $g \in G$ (in which case we say a and b are *conjugates*).

   (a) Prove that $\sim$ is an equivalence relation.

   (b) The equivalence classes of $\sim$ are called *conjugacy classes*. For $a \in G$, prove that a is in the center of G if and only if its conjugacy class is $\{a\}$.

6. Let H be the *quaternion group*, which can be represented as the group of matrices

$$H = \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$$

   where

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathbf{i} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \ \mathbf{j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \mathbf{k} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

   The elements of H satsify the relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

   Find the conjugacy classes of H, and the center of H.

7. (2.8.4) Let G be a group of order 35.

   (a) Prove that G contains an element a of order 5.

   (b) Prove that G contains an element b of order 7.

   (c) Prove that $\langle a, b \rangle = G$.
   [Hint: show that the elements $a^n b^m$ with $0 \leq n < 5$ and $0 \leq m < 7$ are distinct.]