MATH 150A Winter 2020 - **Problem Set 5 solutions**

due February 14

1. Let C_n denote the cyclic group of order n.

   (a) For which pairs of positive integers n and m is $C_n \times C_m$ cyclic?
   As shown on the previous problem set, if $x \in C_n$ has order r and $y \in C_m$ has order s, then $(x, y) \in C_n \times C_m$ has order $\operatorname{lcm}(r, s)$. Note that r must divide n and s must divide m. Therefore the only way that $\operatorname{lcm}(r, s) = nm$ is if $r = n$, $s = m$ and n and m are relatively prime. Therefore $C_n \times C_m$ is cyclic if and only if n and m are relatively prime.

   (b) Prove that $\mathbb{Z} \times \mathbb{Z}$ is not cyclic.
   Let (a, b) be a non-identity element of $\mathbb{Z} \times \mathbb{Z}$. Then
   $$\langle (a, b) \rangle = \{(na, nb) \mid n \in \mathbb{Z}\}.$$
   This set is contained in a line so (a, b) doesn't generate all of $\mathbb{Z} \times \mathbb{Z}$. For example the element $(-b, a)$ is not in this set. Therefore $\mathbb{Z} \times \mathbb{Z}$ is not cyclic.

2. Let G and H be groups and $\varphi : H \to \operatorname{Aut}(G)$ a homomorphism. The *semidirect product group*, $G \rtimes_\varphi H$, is defined as the set $G \times H$ with operation
   $$(g_1, h_1)(g_2, h_2) = (g_1 \varphi(h_1)(g_2), h_1 h_2).$$

   (a) Prove that $G \rtimes_\varphi H$ is a group.
   For readability, denote the automorphism $\varphi(h)$ by φ_h.
   Associativity:
   $$((g_1, h_1)(g_2, h_2))(g_3, h_3) = (g_1 \varphi_{h_1}(g_2), h_1 h_2)(g_3, h_3) = (g_1 \varphi_{h_1}(g_2) \varphi_{h_1 h_2}(g_3), h_1 h_2 h_3),$$
   $$(g_1, h_1)((g_2, h_2)(g_3, h_3)) = (g_1, h_1)(g_2 \varphi_{h_2}(g_3), h_2 h_3) = (g_1 \varphi_{h_1}(g_2 \varphi_{h_2}(g_3)), h_1 h_2 h_3).$$
   Since φ_{h_1} is a homomorphism, $\varphi_{h_1}(g_2 \varphi_{h_2}(g_3)) = \varphi_{h_1}(g_2) \varphi_{h_1}(\varphi_{h_2}(g_3))$, and since φ is itself a homomorphism, $\varphi_{h_1} \circ \varphi_{h_2} = \varphi_{h_1 h_2}$, so the two terms above are equal.
   Identity: The identity element of $G \rtimes_\varphi H$ is $(1, 1)$. For any (g, h),
   $$(1, 1)(g, h) = (1 \cdot \varphi_1(g), 1 \cdot h) = (g, h),$$
   $$(g, h)(1, 1) = (g \cdot \varphi_h(1), h \cdot 1) = (g, h),$$
   following from the facts that φ_1 is the identity map, so $\varphi_1(g) = g$, and that φ_h is a homomorphism, so $\varphi_h(1) = 1$.
   Inverses: For $(g, h) \in G \rtimes_\varphi H$, the inverse is $(\varphi_{h^{-1}}(g^{-1}), h^{-1})$.
   $$(g, h)(\varphi_{h^{-1}}(g^{-1}), h^{-1}) = (g \varphi_h(\varphi_{h^{-1}}(g^{-1})), h h^{-1}) = (g g^{-1}, h h^{-1}) = (1, 1),$$
   $$(\varphi_{h^{-1}}(g^{-1}), h^{-1})(g, h) = (\varphi_{h^{-1}}(g^{-1}) \varphi_{h^{-1}}(g), h^{-1} h) = (\varphi_{h^{-1}}(g^{-1} g), h^{-1} h) = (1, 1),$$
   following from the fact that $\varphi_{h^{-1}}$ is the inverse map of φ_h, so $\varphi_h(\varphi_{h^{-1}}(g^{-1})) = g^{-1}$.

(b) Prove that $G \times \{1\}$ is a normal subgroup of $G \rtimes_\varphi H$.

Let $(g, 1) \in G \times \{1\}$ and $(a, b) \in G \rtimes_\varphi H$.

$$\begin{aligned}(a,b)(g,1)(a,b)^{-1} = (a\varphi_b(g), b)(\varphi_{b^{-1}}(a^{-1}), b^{-1}) &= (a\varphi_b(g)\varphi_b(\varphi_{b^{-1}}(a^{-1})), bb^{-1}) \\ &= (a\varphi_b(g)a^{-1}, 1)\end{aligned}$$

which is in $G \times \{1\}$, so it is normal.

3. Let D_n denote the dihedral group for a regular n-gon with $n \geq 3$. Show that D_n has a semidirect product structure,

$$D_n \cong C_n \rtimes_\varphi C_2.$$

What is $\varphi : C_2 \to \mathrm{Aut}(C_n)$ in this case?

D_n is generated by r and s, a rotation and a reflection. Let $G = \langle r \rangle \cong C_n$ and $H = \langle s \rangle \cong C_2$. Then the elements of $G \times H$ are (r^k, s^ℓ) for $0 \leq k < n$ and $0 \leq \ell < 2$. The elements of D_n have the form $r^k s^\ell$ for $0 \leq k < n$ and $0 \leq \ell < 2$. Define a function $f : C_n \rtimes_\varphi C_2 \to D_n$ by $f(r^k, s^\ell) = r^k s^\ell$. It is clear that f is a bijection. We will show it is also a homomorphism for φ as defined below.

Define $\varphi : C_2 \to \mathrm{Aut}(C_n)$ by $\varphi_1(r^k) = r^k$ and $\varphi_s(r^k) = r^{-k}$. In other words $\varphi_{s^\ell}(r^k) = r^{(-1)^\ell k}$. Then

$$(r^k, s^\ell)(r^p, s^q) = (r^m \varphi_{s^\ell}(r^p), s^\ell s^q) = (r^{k+(-1)^\ell p}, s^{\ell+q}).$$

To show that f is a homomorphism,

$$f(r^k, s^\ell)f(r^p, s^q) = (r^k s^\ell)(r^p s^q) = r^k r^{(-1)^\ell p} s^\ell s^q = f(r^{k+(-1)^\ell p}, s^{\ell+q}).$$

Since f is a bijective homomorphism, $C_n \rtimes_\varphi C_2 \cong D_n$.

4. (7.1.2) Let H be a subgroup of group G. Describe the orbits of the H-action on G by left multiplication.

The H-orbit of $g \in G$ is

$$O_g = \{hg \mid h \in H\} = Hg,$$

so the orbits are the right cosets of H.

5. $\mathrm{O}(n)$ denotes the *orthogonal group*, the subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of all real orthogonal $n \times n$ matrices. These are the rotations and reflections of $\mathbb{R}^n$ that fix the origin. Find the orbits of the $\mathrm{O}(2)$-action on $\mathbb{R}^2$. For a point $(x, y) \in \mathbb{R}^2$ what is its stabilizer?

Orthogonal transformations are the ones that preserve distances. Therefore the orbit of $(a, b) \in \mathbb{R}^2$ is the set of all points at the same distance from the origin. If $\sqrt{a^2 + b^2} = r$ then

$$O_{(a,b)} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\},$$

which is the cicle centered at the origin with radius r.

The stabilizer of $(0, 0)$ is all of $\mathrm{O}(2)$ because the origin is fixed by all linear transformations. For (a, b) that is not the origin, the stabilizer has two elements, the identity map, and the reflection across the line through (a, b),

$$\mathrm{O}(2)_{(a,b)} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \frac{1}{r^2} \begin{bmatrix} a^2 - b^2 & 2ab \\ 2ab & b^2 - a^2 \end{bmatrix} \right\}.$$