MATH 150A Winter 2020 - **Problem Set 6**

due February 24

1. Let G be a group of order n that acts operates non-trivially on a set of size r. Prove that if $n > r!$, then G has a proper normal subgroup. (A *proper* subgroup of G is a subgroup that is neither trivial nor equal to G.)

2. (a) Prove that the transpositions $(1\ 2), (2\ 3), \ldots, (n-1\ n)$ generate the symmetric group S_n.

   (b) How many transpositions are needed to write the cycle $(1\ 2\ 3 \cdots n)$?

   (c) Prove that the cycle $(1\ 2\ 3 \cdots n)$ and $(1\ 2)$ generate the symmetric group S_n.

3. Let σ be the 5-cycle $(1\ 2\ 3\ 4\ 5)$ in S_5. Find the element $\tau \in S_5$ which accomplishes the specified conjugation:

   (a) $\tau\sigma\tau^{-1} = \sigma^2$,

   (b) $\tau\sigma\tau^{-1} = \sigma^{-1}$,

   (c) $\tau\sigma\tau^{-1} = \sigma^{-2}$.

4. Let C be the conjugacy class of an even permutation p in S_n. Show that C is either a conjugacy class in A_n, or else the union of two conjugacy classes in A_n of equal size. Explain how to decide which case occurs in terms of the centralizer of p.

5. Find the class equation for S_6 and give a representative for each conjugacy class.

6. Let G be a group of order 200. Prove that G has a normal Sylow 5-subgroup.

7. Let G be a group of order 105. Prove that G has a proper normal subgroup.