MATH 150A Winter 2020 - **Problem Set 6 solutions**

due February 24

1. Let G be a group of order n that acts operates non-trivially on a set of size r. Prove that if $n > r!$, then G has a proper normal subgroup. (A *proper* subgroup of G is a subgroup that is neither trivial nor equal to G.)

   Let X be the set that G acts on with $|X| = r$ and $|G| = n > r!$. A G action on X defines a homomorphism
   $$\varphi : G \to \operatorname{Perm}(X)$$
   where $\operatorname{Perm}(X)$ denotes the permutation group of X. Since $|\operatorname{Perm}(X)| = r!$, $|G| > |\operatorname{Perm}(X)|$ so the map φ cannot be injective. Therefore $\ker\varphi$ is a nontrivial normal subgroup of G. Since the action of G is non-trivial, $\ker\varphi \neq G$, so $\ker\varphi$ is a proper normal subgroup.

2. (a) Prove that the transpositions $(1\ 2), (2\ 3), \dots, (n-1\ n)$ generate the symmetric group S_n.

   Let H be the subgroup generated by transpositions. We will prove that $H = S_n$.

   Let $c_{a,b} = (a\ a+1\ \cdots\ b-1\ b)$ for $1 \le a < b \le n$. These cycles are in H by
   $$c_{a,b} = (a\ a+1)(a+1\ a+2)\cdots(b-2\ b-1)(b-1\ b).$$
   An arbitrary swap (a, b) is in H by
   $$(a\ b) = c_{a+1,b}^{-1} c_{a,b}$$
   since $c_{a,b}$ moves b to the position of a and shifts the rest up by 1, and then $c_{a+1,b}^{-1}$ moves a (now in position $a+1$) to the position of b and shifts the rest down by 1 back to where they started. The swaps are all the conjugates of the transpositions.

   From this we can get arbitrary cycles of length m. Any cycle γ of length m is a conjugate of $c_{1,m}$ so $\gamma = \sigma c_{1,m}\sigma^{-1}$ for some $\sigma \in S_n$. Therefore
   $$\begin{aligned}\gamma = \sigma c_{1,m}\sigma^{-1} &= (\sigma(1\ 2)\sigma^{-1})(\sigma(2\ 3)\sigma^{-1})\cdots(\sigma(m-1\ m)\sigma^{-1}) \\ &= (\sigma(1)\ \sigma(2))(\sigma(2)\ \sigma(3))\cdots(\sigma(m-1)\ \sigma(m))\end{aligned}$$
   and each $(\sigma(k)\ \sigma(k+1))$ is in H because it is a swap.

   Finally, every permutation can be expressed as a product of cycles so $H = S_n$.

   (b) How many transpositions are needed to write the cycle $(1\ 2\ 3 \cdots n)$?

   I think the minimum is $n-1$:
   $$c_{1,n} = (1\ 2)(2\ 3)\cdots(n-2\ n-1)(n-1\ n),$$
   but I don't have a proof that there is no shorter expression.

(c) Prove that the cycle $(1\ 2\ 3 \cdots n)$ and $(1\ 2)$ generate the symmetric group S_n.

Since S_n is generated by the transpositions by part (a), we just need to show that all transpositions can be generated by $c_{1,n} = (1\ 2\ 3 \cdots n)$ and $(1\ 2)$. Recall that conjugating $(1\ 2)$ by a permutation σ gives

$$\sigma(1\ 2)\sigma^{-1} = (\sigma(1)\ \sigma(2)).$$

We want to get the transposition $(k\ \ k+1)$ this way. Since $c_{1,n}$ shifts all of the elements (except n) up by one, take $\sigma = c_{1,n}^{k-1}$.

$$(k\ k+1) = c_{1,n}^{k-1}(1\ 2)c_{1,n}^{-k+1}.$$

3. Let σ be the 5-cycle $(1\ 2\ 3\ 4\ 5)$ in S_5. Find the element $\tau \in S_5$ which accomplishes the specified conjugation:

(a) $\tau\sigma\tau^{-1} = \sigma^2$,

(b) $\tau\sigma\tau^{-1} = \sigma^{-1}$,

(c) $\tau\sigma\tau^{-1} = \sigma^{-2}$.

Recall that conjugating σ by a permutation τ gives

$$\tau(1\ 2\ 3\ 4\ 5)\tau^{-1} = (\tau(1)\ \tau(2)\ \tau(3)\ \tau(4)\ \tau(5)).$$

Then

$$\sigma^2 = (1\ 3\ 5\ 2\ 4) = \tau(1\ 2\ 3\ 4\ 5)\tau^{-1} = (\tau(1)\ \tau(2)\ \tau(3)\ \tau(4)\ \tau(5))$$

so $\tau(1) = 1$, $\tau(2) = 3$, $\tau(3) = 5$, $\tau(4) = 2$, $\tau(4) = 4$.

$$\sigma^{-1} = (1\ 5\ 4\ 3\ 2) = \tau(1\ 2\ 3\ 4\ 5)\tau^{-1} = (\tau(1)\ \tau(2)\ \tau(3)\ \tau(4)\ \tau(5))$$

so $\tau(1) = 1$, $\tau(2) = 5$, $\tau(3) = 4$, $\tau(4) = 3$, $\tau(4) = 2$.

$$\sigma^{-2} = (1\ 4\ 2\ 5\ 3) = \tau(1\ 2\ 3\ 4\ 5)\tau^{-1} = (\tau(1)\ \tau(2)\ \tau(3)\ \tau(4)\ \tau(5))$$

so $\tau(1) = 1$, $\tau(2) = 4$, $\tau(3) = 2$, $\tau(4) = 5$, $\tau(4) = 3$.

4. Let C be the conjugacy class of an even permutation p in S_n. Show that C is either a conjugacy class in A_n, or else the union of two conjugacy classes in A_n of equal size. Explain how to decide which case occurs in terms of the centralizer of p.

The conjugacy class of p in S_n is

$$C = \{gpg^{-1} \mid g \in S_n\}.$$

Let C' be the conjugacy class of p in A_n,

$$C' = \{gpg^{-1} \mid g \in A_n\}.$$

Since A_n has index 2 in S_n, S_n is the disjoint union of two right cosets $S_n = A_n \cup A_n\sigma$ where σ is any odd permutation in S_n. Let

$$C'' = \{gpg^{-1} \mid g \in A_n\sigma\}$$

so then $C = C' \cup C''$. Let $q = \sigma p \sigma^{-1}$. Since σ and σ^{-1} are both odd and p is even, q is also even so $q \in A_n$. Each $g \in A_n\sigma$ can be expressed as $g = h\sigma$ for some $h \in A_n$, so then

$$gpg^{-1} = h\sigma p \sigma^{-1} h^{-1} = hqh^{-1}.$$

Therefore C'' is the conjugacy class of q in A_n,

$$C'' = \{hqh^{-1} \mid h \in A_n\}.$$

The conjugacy classes in A_n partition A_n so either $C' = C''$ or they are disjoint. If $C' = C''$ then C is a conjugacy class in A_n. Otherwise C is the disjoint union of conjugacy classes C' and C''.

Let K be the centralizer of p in S_n. Suppose there is an odd permutation $\tau \in K$. Then $\tau = h\sigma$ for some $h \in A_n$ and

$$p = \tau p \tau^{-1} = h\sigma p \sigma^{-1} h^{-1} = hqh^{-1},$$

so $p \in C''$. This implies $C' = C''$. Conversely if $p \in C''$ then $p = hqh^{-1} = h\sigma p \sigma^{-1} h^{-1}$ for some $h \in A_n$, so $h\sigma$ is an odd permutation in K. Therefore p has no odd permutations in its centralizer if and only if C' and C'' are disjoint.

Suppose $C' \neq C''$ so there are no odd permutations in K. Group S_n acts on S_n by conjugation and the orbit of p under this action is C, while its stabilizer is K. The counting formula gives

$$|S_n| = |C||K|.$$

Restricting the action to A_n, the orbit of p is C', but the centralizer of p is the same, since $K \subseteq A_n$. The counting formula gives

$$|A_n| = |C'||K|.$$

Since $2|A_n| = |S_n|$, this implies that $2|C'| = |C|$ so then $|C'| = |C''|$.

5. Find the class equation for S_6 and give a representative for each conjugacy class.

   The conjugacy classes of S_6 correspond to the possible cycle structures.

   - The conjugacy class of the identity has 1 element.
   - The conjugacy class of $(1\ 2)$ has $\binom{6}{2} = 15$ elements.
   - The conjugacy class of $(1\ 2\ 3)$ has $2!\binom{6}{3} = 40$ elements.
   - The conjugacy class of $(1\ 2\ 3\ 4)$ has $3!\binom{6}{4} = 90$ elements.
   - The conjugacy class of $(1\ 2\ 3\ 4\ 5)$ has $4!\binom{6}{5} = 144$ elements.
   - The conjugacy class of $(1\ 2\ 3\ 4\ 5\ 6)$ has $5!\binom{6}{6} = 120$ elements.

- The conjugacy class of (1 2)(3 4) has $3\binom{6}{4} = 45$ elements.
- The conjugacy class of (1 2 3)(4 5) has $\binom{6}{5}2\binom{5}{3} = 120$ elements.
- The conjugacy class of (1 2 3 4)(5 6) has $3!\binom{6}{4} = 90$ elements.
- The conjugacy class of (1 2 3)(4 5 6) has $2\binom{6}{3} = 40$ elements.
- The conjugacy class of (1 2)(3 4)(5 6) has $\binom{6}{4}\binom{4}{2}/3! = 15$ elements.

Therefore the class equation is

$$1 + 15 + 40 + 90 + 144 + 120 + 45 + 120 + 90 + 40 + 15 = 720.$$

6. Let G be a group of order 200. Prove that G has a normal Sylow 5-subgroup.

   The number of Sylow 5-subgroups must be $5k + 1$ for an integer k, and it must divide 200. The divisors of 200 are

   $$1, 2, 4, 5, 10, 20, 40, 50, 100, 200$$

   and 1 is the only number on the list of the form $5k + 1$. Therefore there is only 1 Sylow 5-subgroup, so it must be normal.

7. Let G be a group of order 105. Prove that G has a proper normal subgroup.

   The divisors of 105 are

   $$1, 3, 5, 7, 15, 21, 35, 105.$$

   The number of Sylow 5-subgroups is 1 or 21. The number of Sylow 7-subgroups is 1 or 15. The number of Sylow 3-subgroups is 1 or 7. Suppose that G has no proper normal subgroups. Then G has 21 subgroups of order 5, 15 subgroups of order 7 and 7 subgroups of order 3. Each Sylow 5-subgroup has 4 non-trivial elements, each with order 5. Any two distinct Sylow 5-subgroups have trivial intersection. Therefore G has $21 \cdot 4 = 84$ elements of order 5. Similarly G has $15 \cdot 6 = 90$ elements of order 7 and $7 \cdot 2 = 14$ elements of order 3. However G has only 105 elements, so this is a contradiction. Therefore G must have a proper normal subgroup.