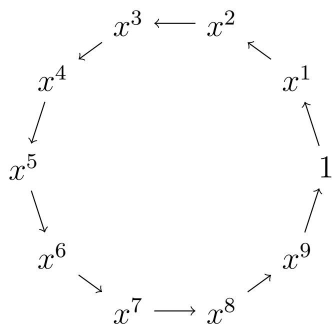
MATH 150A Winter 2020 - **Problem Set 8 solution**

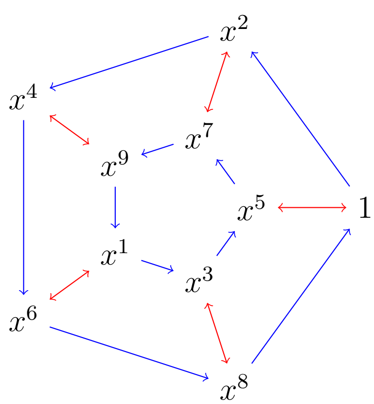
due March 6

1. Draw the Cayley graph for each group and generating set.

   (a) C_{10} generated by $\{x\}$.

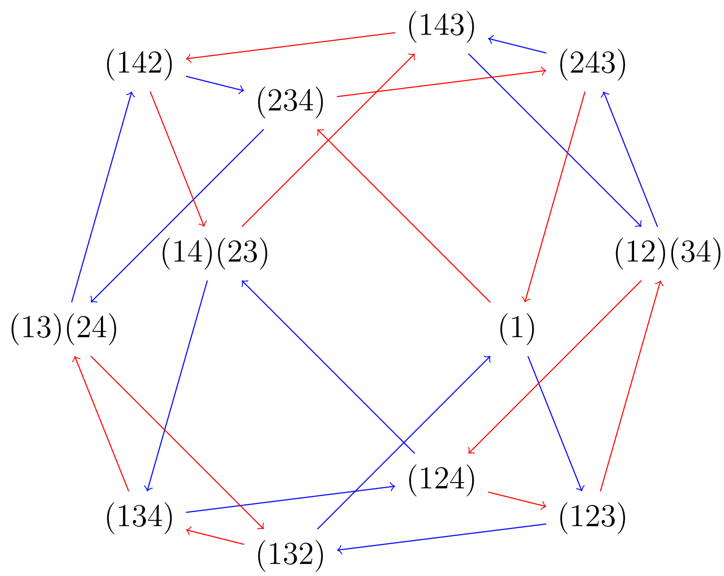


   (b) C_{10} generated by $\{x^2, x^5\}$.



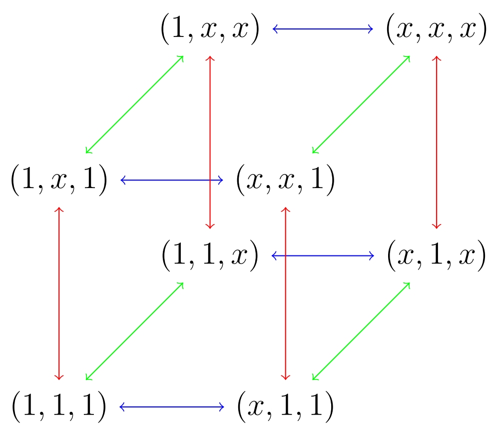
   Edges for x^2 are in blue, and for x^5 are in red.

   (c) A_4 generated by $\{(1\ 2\ 3), (2\ 3\ 4)\}$.



Edges for (1 2 3) are in blue, and for (2 3 4) are in red.

(d) $C_2 \times C_2 \times C_2$ generated by $\{(x, 1, 1), (1, x, 1), (1, 1, x)\}$.



Edges for $(x, 1, 1)$ are in blue, for $(1, x, 1)$ are in red, and for $(1, 1, x)$ are in green.

2. Let G be a group generated by S and H a subgroup of G generated by $T \subseteq S$. Prove that H is normal in G if and only if all edges labelled by elements of T are loops in the Schreier coset graph of H in G with generating set S.

   The vertex set of the Schreier coset graph is the set right-cosets $\{Hg \mid g \in G\}$. If all edges labelled by elements of T are loops in the graph, then $Hgt = Hg$ for all $t \in T$ and $g \in G$. Therefore $gt = hg$ for some $h \in H$, so the conjugation gtg^{-1} is in H for all $t \in T$ and $g \in G$. Since T generates H, this implies that $ghg^{-1} \in H$ for all $h \in H$ and $g \in G$, so H is normal.

   Conversely if H is normal, $ghg^{-1} \in H$ for all $h \in H$ and $g \in G$, so in particular $gtg^{-1} \in H$ for all $t \in T$. Therefore $Hgt = Hg$ so all edges labelled by t are loops.

3. Given two elements of the lamplighter group

$$g = (n, (\ldots, l_{-1}, l_0, l_1, \ldots)),$$

$$h = (m, (\dots, k_{-1}, k_0, k_1, \dots)),$$

how can one determine if they are conjugates?

If g and h are conjugates, then $h = f^{-1}gf$ for some element

$$f = (p, (\dots, j_{-1}, j_0, j_1, \dots))$$

and the inverse of f is

$$f^{-1} = (-p, (\dots, j_{p-1}, j_p, j_{p+1}, \dots)).$$

Composing, we have that $m = p + n - p = n$ and

$$k_i = j_i + l_{i-p} + j_{i-n}$$

for each $i \in \mathbb{Z}$. This gives a telescoping series for each $i = 0, \dots, n-1$:

$$\sum_{a \in \mathbb{Z}} k_{an+i} = \sum_{a \in \mathbb{Z}} l_{an+i-p}.$$

These series converge because only a finite number of summands are non-zero. The conditions for g and h to be conjugates are $n = m$ and that there exists an integer p such that the above series are equal for all $0 \le i < n$. We can also take $0 \le p < n$ since its value only matters modulo n.

4. The *infinite dihedral group* D_∞ is a subgroup of permutations of the integers generated by $f(n) = -n$ and $g(n) = 1 - n$, which reflect the integer number line over the point 0 and 1/2 respectively.

   (a) Give a presentation of D_∞.

   Functions f and g have both $f \circ f$ and $g \circ g$ equal to the identity which gives relations $f^2 = g^2 = 1$. Let

   $$G = \langle f, g \mid f^2, g^2 \rangle.$$

   We want to know if $G = D_\infty$ or if more relations are needed. In G we can reduce any word in $\{f, g, f^{-1}, g^{-1}\}$ to one consisting of alternating f and g, so it has one of the following forms:

   $$fgfg \cdots fg = (fg)^k,$$
   $$fgfg \cdots fgf = (fg)^k f,$$
   $$gfgf \cdots gf = (gk)^k,$$
   $$gfgf \cdots gfg = (gf)^k g$$

   with $k \ge 0$. It can be checked that these all give distnct functions in D_∞ since $(fg)^k(n) = n - k$, $(fg)^k f(n) = -n - k$, $(gf)^k(n) = n + k$ and $(gf)^k g(n) = -n + k + 1$. Therefore $G = D_\infty$.

(b) Demonstrate a surjective homomorphism to each finite dihedral group $\varphi : D_\infty \to D_n$ for $n \geq 3$.

We have that f acts as a reflection of the integer number line, and gf is the function that shifts every integer up by 1 (from part (a)), so let $\varphi(f) = s$ and $\varphi(gf) = r$ for

$$D_n = \langle r, s \mid r^n, s^2, rsrs \rangle.$$

Therefore $\varphi(g) = \varphi(gf \cdot f) = rs$. To check that this is a well-defined homomorphism, each relation in D_∞ must map to the identity.

$$\varphi(f^2) = s^2 = 1,$$

$$\varphi(g^2) = rsrs = 1.$$

The map φ is surjective because a generating set $\{r, s\}$ of D_n is in the image and φ is a homomorphism, so all of D_n is in the image.

5. Use the Todd-Coxeter algorithm to analyze the group generated $\{x, y\}$ with the following relations. Determine the order of the group and identify the group if you can.

(a) $x^2 = 1$, $y^2 = 1$, $xyx = yxy$,

Let $G = \langle x, y \mid x^2, y^2, xyxy^{-1}x^{-1}y^{-1} \rangle$ and $H = \langle x \rangle$.

	x	x
1	1	1
2	3	2
3	2	3

	y	y
1	2	1
2	1	2
3	3	3

	x	y	x	y^{-1}	x^{-1}	y^{-1}
1	1	2	3	3	2	1
2	3	3	2	1	1	2
3	2	1	1	2	3	3

Therefore H has three right-cosets, H, Hy, Hyx represented by 1, 2, 3 respectively. Since $Hy \neq Hyx$, it must be that $1 \neq x$, so the subgroup H generated by x is not trivial. Therefore $|H| = 2$. This means that $|G| = 6$ so it is either C_6 or S_3. If G were abelian then $Hyx = Hxy = Hy$, but this is not the case, so $G = S_3$.

(b) $x^3 = 1$, $y^3 = 1$, $xyx = yxy$,

Let $G = \langle x, y \mid x^3, y^3, xyxy^{-1}x^{-1}y^{-1} \rangle$ and $H = \langle x \rangle$.

	x	x	x
1	1	1	1
2	3	4	2
3	4	2	3
4	2	3	4
5	7	8	5
6	6	6	6
7	8	5	7
8	5	7	8

	y	y	y
1	2	5	1
2	5	1	2
3	3	3	3
4	6	7	4
5	1	2	5
6	7	4	6
7	4	6	7
8	8	8	8

x	y	x	y^{-1}	x^{-1}	y^{-1}	
1	1	2	3	3	2	1
2	3	3	4	7	5	2
3	4	6	6	4	3	3
4	2	5	7	6	6	4
5	7	4	2	1	1	5
6	6	7	8	8	7	6
7	8	8	5	2	4	7
8	5	1	1	5	8	8

Therefore H has 8 right-cosets, H, Hy, Hyx, Hyx^2, Hy^2, Hyx^2y, Hy^2x, Hy^2x^2 represented by 1, 2, 3, 4, 5, 6, 7, 8 respectively. Since $Hy \neq Hyx$, it must be that $1 \neq x$, so the subgroup H generated by x is not trivial. Therefore $|H| = 3$ so $|G| = 24$. $Hy \neq Hyx$ also implies G is not abelian. But $G \neq S_4$ since S_4 is not generated by its order 3 elements.

(c) $x^4 = 1$, $y^2 = 1$, $xyx = yxy$,

Let $G = \langle x, y \mid x^4, y^2, xyxy^{-1}x^{-1}y^{-1} \rangle$ and $H = \langle y \rangle$.

x	x	x	x	
1	2	1	2	1
2	1	2	1	2
3	3	3	3	3

y	y	
1	1	1
2	3	2
3	2	3

x	y	x	y^{-1}	x^{-1}	y^{-1}	
1	2	3	3	2	1	1
2	1	1	2	3	3	2
3	3	2	1	1	2	3

Therefore H has 3 right-cosets, H, Hx, Hxy represented by 1, 2, 3 respectively. Since $Hx \neq Hxy$, it must be that $1 \neq y$, so the subgroup H generated by y is not trivial. Therefore $|H| = 2$ so $|G| = 6$. $Hy \neq Hyx$ also implies G is not abelian, so $G = S_3$. Even though we have the relation $x^4 = 1$, in this group x actually has order 2.

(d) $x^4 = 1$, $y^4 = 1$, $x^2y^2 = 1$,

Let $G = \langle x, y \mid x^4, y^4, x^2y^2 \rangle$. This group is infinite. Its relations are generated by the relations of the infinite dihedral group, $\langle x, y \mid x^2, y^2 \rangle$. We can still analyze G if we choose H to be something large enough that the index $[G : H]$ is finite.

(e) $x^3 = 1$, $y^2 = 1$, $yxyxy = 1$,

Let $G = \langle x, y \mid x^3, y^2, yxyxy \rangle$ and $H = \{1\}$.

x	x	x	
1	1	1	1

y	y	
1	1	1

y	x	y	x	y	
1	1	1	1	1	1

Therefore G is the trivial group.

(f) $x^3 = 1$, $y^3 = 1$, $yxyxy = 1$.

Let $G = \langle x, y \mid x^3, y^3, yxyxy \rangle$ and $H = \{1\}$.

	x	x	x
1	1	1	1
2	2	2	2
3	3	3	3

	y	y	y
1	2	3	1
2	3	1	2
3	1	2	3

	y	x	y	x	y
1	2	2	3	3	1
2	3	3	1	1	2
3	1	1	2	2	3

Therefore G has 3 elements, $\{1, y, y^2\}$ and $x = 1$, so $G = C_3$.