MATH 150A Winter 2020 - **Problem Set 9 solutions**

due March 13

1. Let m be an orientation-reversing isometry of $\mathbb{R}^2$. Prove algebraically that m^2 is a translation.

   An isometry of $\mathbb{R}^2$ can be decomposed into a translation t_a and an orthogonal linear transformation s, so $m = t_a s$. Since m is orientation-reversing, the linear transformation s is a reflection across a line through the origin. Then

   $$m^2 = t_a s t_a s.$$

   We showed in class that for s a linear transformation, $st_a = t_{s(a)}s$. Therefore

   $$m^2 = t_a s t_a s = t_a t_{s(a)} s^2 = t_{a+s(a)}$$

   since $s^2 = 1$.

2. Find the conjugacy class of an isometry of $\mathbb{R}^2$ of each of the following types.

   (a) Translation.

   Let t_a be a translation, and $m = t_b\varphi$ be an arbitrary isometry of $\mathbb{R}^2$ where φ is an orthogonal linear transformation. Then

   $$m t_a m^{-1} = t_b \varphi t_a \varphi^{-1} t_b^{-1}.$$

   The inverse of t_b is t_{-b}. As in Problem 1, We have $\varphi t_a = t_{\varphi(a)}\varphi$ so

   $$t_b \varphi t_a \varphi^{-1} t_b^{-1} = t_b t_{\varphi(a)} \varphi \varphi^{-1} t_{-b} = t_b t_{\varphi(a)} t_{-b} = t_{\varphi(a)}.$$

   Note that $\varphi(a)$ can be any point in $\mathbb{R}^2$ with $|\varphi(a)| = |a|$, so the conjugacy class of t_a is

   $$\{t_c \mid c \in \mathbb{R}^2 \text{ with } |c| = |a|\}.$$

   (b) Rotation about a point.

   Let $t_a\rho_\theta$ be a rotation, and $m = t_b\varphi$ be an arbitrary isometry of $\mathbb{R}^2$. Then

   $$m t_a \rho_\theta m^{-1} = t_b \varphi t_a \rho_\theta \varphi^{-1} t_b^{-1}.$$

   Conjugating t_a by φ gives $t_{\varphi(a)}$. If φ is a rotation, then it commutes with ρ_θ, so $\varphi\rho_\theta\varphi^{-1} = \rho_\theta$. In this case we have

   $$t_b \varphi t_a \rho_\theta \varphi^{-1} t_b^{-1} = t_b t_{\varphi(a)} \rho_\theta t_{-b} = t_b t_{\varphi(a)} t_{b'} \rho_\theta = t_{b+\varphi(a)+b'} \rho_\theta$$

   where $b' = \rho_\theta(-b)$. If φ is a reflection, then $\varphi\rho_\theta\varphi^{-1} = \rho_{-\theta}$, the rotation in the opposite direction. Then

   $$t_b \varphi t_a \rho_\theta \varphi^{-1} t_b^{-1} = t_b t_{\varphi(a)} \rho_{-\theta} t_{-b} = t_b t_{\varphi(a)} t_{b'} \rho_{-\theta} = t_{b+\varphi(a)+b'} \rho_{-\theta}$$

   where $b' = \rho_{-\theta}(-b)$. Assuming $\theta \neq 0$, $b + b'$ can be any vector in $\mathbb{R}^2$ for the right choice of b. Therefore the conjugacy class of ρ_θ is

   $$\{t_c\rho_\theta \mid c \in \mathbb{R}^2\} \cup \{t_c\rho_{-\theta} \mid c \in \mathbb{R}^2\}.$$

(c) Reflection across a line.

Let $t_a r$ be a reflection where r is a reflection across a line through the origin orthogonal to a, and $m = t_b\varphi$ be an arbitrary isometry of $\mathbb{R}^2$. Then

$$mt_arm^{-1} = t_b\varphi t_a r\varphi^{-1}t_b^{-1}.$$

Conjugating r by φ gives another reflection r' across a line through the origin, so

$$t_b\varphi t_a r\varphi^{-1}t_b^{-1} = t_bt_{\varphi(a)}r't_{-b} = t_bt_{\varphi(a)}t_{b'}r' = t_{b+\varphi(a)+b'}r'$$

where $b' = r'(-b)$. The vector $b + b'$ is orthogonal to the reflection line of r' and so is $\varphi(a)$. Therefore the conjugacy class of t_ar is the set of all reflections.

(d) Glide reflection across a line.

Let t_ct_ar be a glide reflection where r is a reflection across a line through the origin orthogonal to a, and c is parallel to the line. Let $m = t_b\varphi$ be an arbitrary isometry of $\mathbb{R}^2$. Then

$$mt_arm^{-1} = t_b\varphi t_bt_ar\varphi^{-1}t_b^{-1}.$$

This works out the same as the previous case

$$t_b\varphi t_ct_ar\varphi^{-1}t_b^{-1} = t_bt_{\varphi(c)+\varphi(a)}r't_{-b} = t_bt_{\varphi(c)+\varphi(a)}t_{b'}r' = t_{b+\varphi(c)+\varphi(a)+b'}r'$$

where $b' = r'(-b)$. The vector $b + b'$ is orthogonal to the reflection line of r' and so is $\varphi(a)$. $\varphi(c)$ is still parallel to the reflection line with $|\varphi(c)| = |c|$. Therefore the conjugacy class is the set of glide reflections with glide of the same distance.

3. Let ℓ_1 and ℓ_2 be lines through the origin in $\mathbb{R}^2$ that intersect at an angle of π/n and let r_i be the reflection across ℓ_i. Prove that r_1 and r_2 generate a dihedral group D_n.

Suppose ℓ_1 is at angle θ from horizontal, and ℓ_2 is at angle $\theta+\pi/n$. Let u be the reflection across the x-axis. Then r_1 and r_2 can be expressed as

$$r_1 = \rho_{2\theta}u \quad \text{and} \quad r_2 = \rho_{2\theta+2\pi/n}u.$$

The conjugation of a rotation by a reflection is equal to the rotation in the opposite direction, so $u\rho_{2\theta}u^{-1} = \rho_{-2\theta}$. Note also that $u = u^{-1}$. Therefore

$$r_2r_1 = \rho_{2\theta+2\pi/n}u\rho_{2\theta}u = \rho_{2\theta+2\pi/n}\rho_{-2\theta} = \rho_{2\pi/n}.$$

A reflection and a rotation by angle $2\pi/n$ generate the dihedral group D_n, so r_1 and r_2r_1 generate D_n. The group generated by r_1 and r_2r_1 is also the group generated by r_1 and r_2 since $r_2 = r_2r_1 \cdot r_1$.

4. Let S and S' be subsets of $\mathbb{R}^n$. S is *dense* in S' if for every point $a \in S'$ and every $\varepsilon > 0$, there is $s \in S$ with $|a - s| < \varepsilon$.

(a) Prove that an additive subgroup G of $\mathbb{R}$ is either dense in $\mathbb{R}$ or else discrete.

Suppose that G is not discrete, so for any $\epsilon > 0$, there exist $x, y \in G$ with $x \neq y$ such that $|x - y| < \epsilon$. Since G is a subgroup, $x - y$ and $y - x$ are also in G, so there is some element $b \in G$ with $0 < b < \epsilon$. For any real number $a \in \mathbb{R}$, there is an integer n such that $nb \leq a < (n + 1)b$ by taking $n = \lfloor a/b \rfloor$. Then $nb \in G$ and $|a - nb| < b < \epsilon$. Therefore G is dense in $\mathbb{R}$.

(b) Prove that the additive subgroup of $\mathbb{R}$ generated by 1 and $\sqrt{2}$ is dense in $\mathbb{R}$.

Let this subgroup be G and suppose it is not dense. By part (a) it must be discrete. We showed in class that a discrete subgroup of $\mathbb{R}$ is trivial or $G = a\mathbb{Z}$ for some real number $a > 0$. Since G contains 1 and $\sqrt{2}$, it is not trivial. Then $1 = na$ and $\sqrt{2} = ma$ for some integers n and m, which means that $a = 1/n$ and that $\sqrt{2} = m/n$. This is a contradiction because $\sqrt{2}$ is irrational. Therefore G must be dense.

(c) Let H be a subgroup of SO_2. Prove that either H is cyclic or dense in SO_2.

SO_2 is the group of rotations of SO_2. Let

$$G = \{\theta \in \mathbb{R} \mid \rho_\theta \in H\}$$

which is an additive subgroup of $\mathbb{R}$. Therefore G is dense or discrete. If G is dense then H is dense. If G is not dense, then G is trivial or G is generated by some real number $a > 0$. In this case H is also trivial or generated by ρ_a, so it is cyclic.

5. Find the symmetry group of

(a) an I-beam, which one can think of as the product set of the letter I and an interval.

The I-beam can be reflected across each of the three coordinate planes. These generate a group of order 8 isomorphic to $C_2 \times C_2 \times C_2$ with elements

$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}.$$

(b) a baseball (or equivalently a tennis ball) accounting for the seam.

A baseball is covered by two pieces of leather stiched together. The rotational symmetry of the ball is generated by a rotation by angle π that turns around each leather piece, and another rotation by angle π that switches the two pieces. Since these generators both have order 2, the rotational group is the Klein four group, $C_2 \times C_2$.

The baseball also has orientation-reversing symmetry, so the order of the full symmetry group is 8. Since these symmetries also have order 2, the group is also isomorphic to $C_2 \times C_2 \times C_2$.